Symmetries of Unimodal Singularities and Complex Hyperbolic Reflection Groups

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Abstract

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In search of discrete complex hyperbolic reflection groups in a singularity context, we consider cyclic symmetries of the 14 exceptional unimodal function singularities. In the 3-variable case, we classify all the symmetries for which the restriction of the intersection form of an invariant Milnor fibre to a character subspace has positive signature 1, and hence the corresponding equivariant monodromy is a reflection subgroup of $U(k-1,1)$. For such subspaces, we construct distinguished vanishing bases and their Dynkin diagrams. For $k=2$, the projectivised hyperbolic monodromy is a triangle group of the Poincaré disk. For $k=3$, we identify some of the projectivised monodromy groups within a recently published survey by J. R. Parker.

for Ian Porteous

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Chapter 1

Introduction

One of the most famous classical results in singularity theory is the Arnold and Brieskorn discovery of the close relationship between simple function singularities and Weyl groups A_μ, D_μ, E_μ [1, 6]. A few years after it, Arnold extended the relationship to simple singularities with the $\mathbb{Z}_2$ reflection symmetry and Weyl groups B_μ, C_μ, F_4.

Consideration of $\mathbb{Z}_m$ symmetries of simple functions led in [11, 12, 13, 22] to the appearance of Shephard-Todd groups in function singularities. The emphasis there was on realisations of the complex reflection groups as equivariant monodromy groups acting on the appropriate character subspaces in the homology of invariant Milnor fibres, and on the coincidence of the discriminants of the reflection groups and of the $\mathbb{Z}_m$-equivariant functions.

A further series of papers [14, 15, 16], on cyclic symmetries of the parabolic functions, brought in similar singularity realisations of certain complex crystallographic groups [20].

In this thesis, we are naturally expanding the programme to cyclic symmetries of the 14 exceptional unimodal function singularities on one hand, and complex hyperbolic reflection groups on the other. The basic idea is as follows. In the 3-variable case, the intersection form on the vanishing homology of an exceptional unimodal function f is non-degenerate and has positive signature 2. Assume g is an automorphism of $\mathbb{C}^3$ of finite order m, and our function is g-invariant. Then g acts on the second homology of the Milnor fibre $f^{-1}(\varepsilon)$, and decomposes it into a direct sum of the character subspaces H_χ, $\chi^m = 1$, on which g acts as multiplication by χ. Assume the rank 2 positive subspace of the intersection form splits between two character summands. Then we discover that the monodromy within a g-invariant versal deformation of f acts as a complex hyperbolic reflection group on each of the two summands. Developing further the technique introduced in papers on cyclically symmetric functions [11, 12, 13], we can construct vanishing bases in the hyperbolic summands and obtain the generating reflections as the corresponding Picard-Lefschetz operators.

The main result of the thesis is a complete classification of the symmetries of the 14 singularities, which split the positive subspace in the vanishing homology, and the description of the complex hyperbolic groups arising. The latter is done via construction of the distinguished vanishing bases in the relevant character subspaces, and via calculation of the Dynkin diagrams for such bases. All the rank 2 reflection groups obtained projectivise to the triangle groups of the Poincaré disk, while projectivisations of some of our rank 3 groups may be found in [18].

It should be noted that it is the first time when complex hyperbolic reflection groups are appearing in a singularity theory context.

The thesis is organised as follows. Chapter 2 introduces the notion of singularities with symmetry, recalls the definitions and constructions given in [11, 12, 13], and states the exact problem we are solving in this thesis. In Chapter 3 we give an exposition of results coming from previous papers (those already mentioned as well as [14, 15, 16]), for ease of reference when considering the later chapters. The classifications for the invariant and equivariant problems are given in Chapter 4. Finally, Chapter 5 discusses properties of the monodromy groups and identifies some of the low dimensional groups.

The author wishes to acknowledge useful discussions with John Parker and Anna Pratoussevitch on complex hyperbolic reflection groups, and continued help and support from the thesis supervisor Victor Goryunov. Thanks also to Fawaz Alharbi, Graham Reeve and Öykü Yurttas for a friendly and lively atmosphere in our postgraduate student office.

Chapter 2

Singularities with Symmetries

2.1 Equivariant monodromy

2.1.1 Symmetries and deformations

Our main object of study will be pairs (f, g) consisting of a holomorphic function germ $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ with an isolated singularity of right equivalence class X (denoted $X \ni f$), and a finite order automorphism g of $(\mathbb{C}^{n+1}, 0)$ such that $f \circ g = \kappa f$, for some constant $\kappa \in \mathbb{C}$. The automorphism will be called either an *invariant symmetry* of the function when $\kappa = 1$, or an *equivariant symmetry* of the function when $\kappa \neq 1$. We will not distinguish between pairs (g, κ) generating the same cyclic group acting on $\mathbb{C}^{n+1} \oplus \mathbb{C}$.

Since the automorphism g is constrained by f, we may identify g with a linear map $A \in GL(n+1, \mathbb{C})$. This map may be diagonalised $D = P^{-1}AP$, where matrices P^{-1}, P may be viewed as diffeomorphic coordinate changes on $(\mathbb{C}^{n+1}, 0)$ thus preserving the right equivalence class of f.

Assume the coordinates $x_0, \ldots, x_n$ in $(\mathbb{C}^{n+1}, 0)$ are chosen so that g is diagonal according to the previous paragraph. Consider a deformation

$$F_g = f + \sum_{i=1}^{k} \lambda_i \varphi_i \tag{2.1}$$

of the function f, where the λ_i are parameters, and $\{\varphi_1, \ldots, \varphi_k\}$ is the set of all g-equivariant (i.e. $\varphi_i \circ g = \kappa\varphi_i$ for all i) elements of a monomial basis of the local ring Q_f of f,

$$Q_f = \frac{\mathbb{C}\left[x_0, \ldots, x_n\right]}{\left\langle \frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle}.$$

In the standard sense, the deformation F is a g-versal deformation of f. We refer to F_g as an *invariant* or *equivariant deformation* of f (depending on context).

All through the thesis, we use notation ε_m for $e^{2\pi i/m}$, and reserve ω for ε_3.

Example 2.1. Let f be a quasihomogeneous function of degree N with respect to the weights $w_0, \ldots, w_n \in \mathbb{N}$ of the coordinates x_j on $\mathbb{C}^{n+1}$. Assume $gcd(w_0, \ldots, w_n) = 1$, in which case N is the *Coxeter number* of f. Consider the *Coxeter transformation*

$$C: \quad x_j \mapsto \varepsilon_N^{w_j} x_j\,, \qquad j = 0, \ldots, n,$$

of $\mathbb{C}^{n+1}$. This is an invariant symmetry of f. The Coxeter element C generates a discrete subgroup of $U(1)$ corresponding to the values of f making one full anti-clockwise rotation in $\mathbb{C}$ about the origin. Take for an invariant symmetry g of f a power of C that has order m: $g = C^p$, $g^m = id$. Then the φ_i in (2.1) are exactly those elements of a monomial basis of Q_f whose degrees are divisible by m.

In the equivariant case, suppose the deformation monomial of minimal degree φ' has degree d'. Then the symmetry g on the meromorphic function f/φ' is invariant, and the φ_i in 2.1 are exactly those elements of a monomial basis of Q_f whose degrees d are such that $(d - d')$ is divisible by m. This generalisation also holds for the invariant case since the constant monomial is of degree 0 (cf. [22] and [23]).

Example 2.2. Consider the singularity E_{12} with normal form $f = x^3+y^7+z^2$ with the invariant symmetry $g(x, y, z) = (x, \varepsilon_7 y, z)$. The order of g is 7. The monomials preserved by this symmetry are $1, x$ of weight $0, 14$ respectively.

These are the only elements of a monomial basis for Q_f with weights divisible by 7 (see Table 2.1 on page 11).

We shall use the notation Λ for the base $\mathbb{C}^k$ with coordinates $\lambda_1, \ldots, \lambda_k$ of a g-miniversal deformation (2.1) of a function f.

Definition 2.3. The *discriminant* $\Sigma \subset \Lambda$ of f is the set of all values $\lambda \in \Lambda$ of the parameters for which the members of the family F_g have critical value 0.

For a versal deformation (i.e. take g to be the identity) the deformation F_g at a generic point $\lambda \in \Lambda$ has no critical point with value 0 and its derivatives are smooth. This is not necessarily true when g is not equal to the identity map, and the following definition is required in order to maintain this generic behaviour.

Definition 2.4. A deformation F_g is said to be *smoothable* if the discriminant $\Sigma \subset \Lambda$ is a hypersurface.

Since for an invariant deformation a non-zero constant function must be among the φ_i, the deformation is smoothable.

Proposition 2.5. *If an equivariant deformation is smoothable, then at least one of the φ_i must be linear.*

Proof. Assume otherwise. Then the φ_i of lowest degree must be at least quadratic, and for any $\lambda \in \Lambda$ there is a critical point at the origin of $\mathbb{C}^{n+1}$ on the zero level. $\square$

In what follows we will be working with representatives of germs of functions and sets we have introduced, but we will be still denoting them by the same letters.

2.1.2 Roots of the Coxeter transformation

For a singularity $X \ni f$, let C be the Coxeter transformation of order N given in coordinates by

$$C(x_0, \ldots; f) = (\varepsilon_a^\alpha x_0, \ldots; f), \quad (\alpha, a) = 1, \quad \alpha, a \in \mathbb{Z}$$

where $\alpha/a = w_{x_0}/N$ is the weight of x_0, normalised so that the weight of f is $w_f = 1$. Take the canonical choice for the root

$$(\varepsilon_a)^{\frac{1}{s}} := \varepsilon_{as},$$

which we apply to each coordinate in turn to define the map

$$C^{\frac{1}{s}}(x_0, \ldots; f) = (\varepsilon_{as}^{\alpha} x_0, \ldots; \varepsilon_s f).$$

The map $C^{\frac{1}{s}}$ and each of its powers gives an equivariant symmetry of f. Consider the b^{th} power

$$C^{\frac{b}{s}} = (\varepsilon_{as}^{\alpha b} x, \ldots, \varepsilon_s^b f),$$

where we may choose b such that $\gcd(b, s) = 1$, since we otherwise would have chosen a different value for s.

By proposition 2.5, any smoothable equivariant deformation has a linear term in its deformation. Assume without loss of generality that this is the monomial x_0. Then

$$\varepsilon_{as}^{\alpha b} = \varepsilon_s^b.$$

For this equality, we must choose b such that $a|b$. Since N is the Coxeter number, we generate all possible non-equivalent equivariant symmetries by letting b run through all divisors of N such that if b_i and b_j are two choices for b, then b_i and b_j have pairwise distinct greatest common divisors with N for all $i \neq j$. We may then find other equivariant symmetries by changing n. The order of the symmetry is $m = sN/b$.

2.1.3 Basic equivariants

Definition 2.6. If g is an equivariant symmetry of the function f such that the monomial x_0 appears in the g-versal deformation, then it is an invariant symmetry of the meromorphic function f/x_0. We choose g_{x_0} to be a symmetry of this type of highest order, and call g_{x_0} the *basic equivariant of f with respect to x_0.*

If we find $g_x = C^{\frac{b}{s}}$, or this composed with an invariant symmetry, for some

b, s, then we may use the argument above to classify all equivariant deformations of f preserving x_0.

Most of the invariant symmetries that will appear in our consideration are the involutions

$$\iota_I : x_i \mapsto \begin{cases} x_i, & x_i \notin I \\ -x_i, & x_i \in I \end{cases}$$

where $I \subseteq \{x_0, \ldots, x_n\}$ is a subset of the coordinates. E.g.

$$\iota_x(x, y, z) = (-x, y, z).$$

Example 2.7. For $E_{12} \ni f = x^3 + y^7 + z^2$, the Coxeter transformation is given by

$$C(x, y, z; f) = (\omega x, \varepsilon_7 y, -z; f).$$

The basic equivariant g_x in an invariant symmetry of highest order of the meromorphic function $f/x = x^2 + y^7/x + z^2/x$. We may take

$$g_x(x, y, z; f) = (-x, \varepsilon_{14}^3 y, -iz; -f),$$

and we have

$$g_x = C^{\frac{3}{2}}.$$

2.1.4 Symmetric Milnor fibre and its monodromy

To define a Milnor fibre of the function germ f with isolated singularity, we follow the usual procedure. Namely, we take a closed ball $B \subset \mathbb{C}^{n+1}$ of sufficiently small radius and centred at the origin, and assume that the deformation base Λ is a very small ball. Then a Milnor fibre V_λ is $F_\lambda^{-1}(0) \cap B$ provided λ is non-discriminantal ($\lambda \in \Lambda \setminus \Sigma$). In good cases, for example when the function is quasihomogeneous with positive weights, we may expand our deformation representative from the product of the balls to the whole $\mathbb{C}^{n+1} \times \Lambda$. For more details see [5], Sections 1.7 and 1.10.

Let us fix a generic point $\star \in \Lambda \setminus \Sigma$. The Milnor fibre $V_\star$ is homotopic to a wedge of μ n-spheres, where μ is the Milnor number of f. A symmetry g

sends $V_\star$ into itself. Therefore, its nth homology, of total rank μ, is a direct sum of *character subspaces*

$$H_n(V_\star, \mathbb{C}) = \oplus_{\chi^m=1} H_\chi\,, \tag{2.2}$$

where m is the order of the automorphism g, and g acts as multiplication by χ on H_χ.

There is a standard way to define elements of the H_χ analogous to the ordinary Morse vanishing cycles. Namely, let W be the quotient of the fibre $V_\star$ by the action of the group $\mathbb{Z}_m$ generated by g, and $W' \subset W$ its subset of irregular orbits. Since all functions F_λ in the family F are g-invariant (or equivariant depending on context), a path in $\Lambda \setminus \Sigma$ from the point $\star$ to a generic point of the discriminant defines - at least in all our cases - a vanishing cycle $\sigma \in H_n(W, W'; \mathbb{Z})$: similar to what has been observed and used in [2, 11, 12, 13], in all our cases it is easy to see a generator of the relative integer homology which contracts to a point on the approach to the discriminant. The inverse image of this relative cycle in $V_\star$ consists of m chains $\sigma_0 \ldots, \sigma_{m-1}$, with the orientation inherited from σ, and ordered in the cyclic way:

$$g(\sigma_i) = \sigma_{(i+1)\,\mathrm{mod}m}.$$

For appropriate values of χ (the notation $\overline{\chi}$ denotes its conjugate), and in all the cases which will follow, the linear combination

$$\sigma_\chi = \sum_{i=0}^{m-1} \overline{\chi}^i \sigma_i$$

is a cycle, and thus provides an element of H_χ. We call σ_χ a *vanishing χ-cycle.* See Figure 2.1. Examples of these calculations are given in Chapter 3.
The monodromy representation of the fundamental group $\pi_1(\Lambda \setminus \Sigma, \star)$ on $H_n(V_\star, \mathbb{C})$ is a direct sum of the representations on the individual summands H_χ. We denote the corresponding monodromy groups M_χ.

Depending on the parity of n, the intersection form on $H_n(V_\star, \mathbb{Z})$ naturally extends to $H_n(V_\star, \mathbb{C})$ in either an Hermitian or skew-Hermitian way. Assume

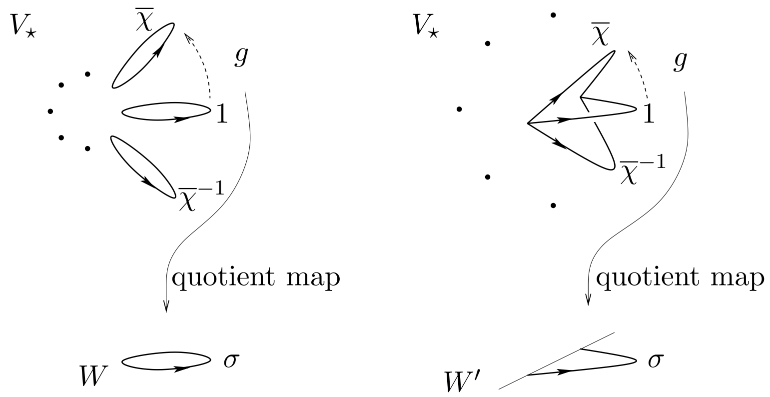


Figure 2.1: Diagram showing χ-cycles as cyclic covers of chains.

that a vanishing χ-cycle σ_χ has a non-zero self-intersection number $\langle\sigma_\chi, \sigma_\chi\rangle$. Then according to [4, 8, 11] the related Picard-Lefschetz operator in M_χ may be brought into the form

$$h_\chi : \quad c \mapsto c + (e-1)\frac{\langle c, \sigma_\chi\rangle}{\langle\sigma_\chi, \sigma_\chi\rangle}\sigma_\chi,$$

where e is the eigenvalue of the operator on σ_χ. This is a (skew-)Hermitian reflection on H_χ.

To obtain a generating set of the M_χ, we proceed in the traditional manner. For this, we start with a generic line $L \subset \Lambda$ passing through the base point $\star$. Let $c_1, \ldots, c_r$ be the points at which L meets Σ. We choose a *distinguished system of paths* on L, that is, paths $\gamma_1, \ldots, \gamma_r$ in L, starting at $\star$ and leading to the c_i, which have no self- and mutual intersections except for the point $\star$ itself. The Picard-Lefschetz operators $h_{i,\chi}$ on the H_χ corresponding to the paths of the system generate M_χ. Thus knowledge of the eigenvalues of the $h_{i,\chi}$ and of the intersection numbers of the χ-cycles vanishing at $c_1, \ldots, c_r$ yields a description of the monodromy group M_χ.

2.2 The exceptional unimodal functions

Now assume that f is one of the 14 exceptional unimodal singularities and $n = 2$ (see the table on page 185 of [3]). Table 2.1 gives a normal form of the quasihomogeneous member of each of the 14 one-parameter families, along with the weights of the coordinates, the Coxeter number N, a monomial basis of the local ring and the weights of its elements. The subscript of the singularity right equivalence type it the Milnor number μ of the singularity. Pairs of functions with the same Coxeter number are dual in the sense of Arnold. Any function with $\mu = 12$ is self-dual.

An arbitrary member of a unimodal family is obtained by addition to the normal form of a multiple of its Hessian, that is, of a multiple of the versal monomial of top weight.

Assume we have two coordinate spaces, $\mathbb{C}^p_{u_1,\dots,u_p}$ and $\mathbb{C}^q_{v_1,\dots,v_q}$, with coordinates of positive integer weights $a_1, \dots, a_p$ and $b_1, \dots, b_q$. Then the space of map-germs from $(\mathbb{C}^p, 0)$ to $(\mathbb{C}^q, 0)$ has a natural grading: a monomial summand $u_1^{\alpha_1} \dots u_p^{\alpha_p}$ in the jth coordinate function is assigned grading $\alpha_1 a_1 + \dots + \alpha_p a_p - b_j$. For example, a quasihomogeneous automorphism g of $\mathbb{C}^p$ has all its monomial terms of grading 0. The determinant $Jac(g)$ of the Jacobi matrix of such automorphism is a non-zero constant, which is easily seen if the coordinates are ordered by the increase of their weights.

In what follows, we are restricting our attention to quasihomogeneous symmetries of exceptional unimodal singularities.

2.2.1 Classification of splitting invariant symmetries

For each of the 14 singularities, the Hermitian intersection form on $H_2(V_\star, \mathbb{C})$ is non-degenerate with positive signature 2. Our aim set in the introduction is to obtain equivariant monodromy groups M_χ which are hyperbolic reflection groups, that is, the restriction of the intersection form to the summand H_χ is non-degenerate and of positive signature 1. Hence the rank 2 positive subspace $H_+ \subset H_2(V_\star, \mathbb{C})$ must split between two character subspaces corresponding to a pair of distinct characters. We refer to a symmetry satisfying this condition as a *splitting symmetry*, and to the two characters as the *hyperbolic characters*. We will use this terminology even in the extreme situation, when the two H_χ are one-dimensional.

Table 2.1: List of unimodal singularities

Type and Normal Form	w_x	w_y	w_z	N	Versal monomials and their weights													
E_{12}	14	6	21	42	1	y	y^2	x	y^3	xy	y^4	xy^2	y^5	xy^3	xy^4	xy^5		
$x^3+y^7+z^2$					0	6	12	14	18	20	24	26	30	32	38	44		
Z_{11}	8	6	15	30	1	y	x	y^2	xy	x^2	y^3	xy^2	y^4	xy^3	xy^4			
$x^3y+y^5+z^2$					0	6	8	12	14	16	18	20	24	26	32			
E_{13}	10	4	15	30	1	y	y^2	x	y^3	xy	y^4	xy^2	y^5	xy^3	y^6	y^7	y^8	
$x^3+xy^5+z^2$					0	4	8	10	12	14	16	18	20	22	24	28	32	
Q_{10}	9	8	6	24	1	z	y	x	z^2	yz	xy	z^3	yz^2	yz^3				
$x^2z+y^3+z^4$					0	6	8	9	12	14	17	18	20	26				
E_{14}	8	3	12	24	1	y	y^2	x	y^3	xy	y^4	xy^2	y^5	xy^3	y^6	xy^4	xy^5	xy^6
$x^3+y^8+z^2$					0	3	6	8	9	11	12	14	15	17	18	20	23	26
Z_{12}	6	4	11	22	1	y	x	y^2	xy	x^2	y^3	xy^2	y^4	x^3	y^5	y^6		
$x^3y+xy^4+z^2$					0	4	6	8	10	12	12	14	16	18	20	24		
W_{12}	5	4	10	20	1	y	x	y^2	xy	x^2	y^3	xy^2	x^2y	xy^3	x^2y^2	x^2y^3		
$x^4+y^5+z^2$					0	4	5	8	9	10	12	13	14	17	18	22		
Q_{11}	7	6	4	18	1	z	y	x	z^2	yz	z^3	xy	yz^2	z^4	z^5			
$x^2z+y^3+yz^3$					0	4	6	7	8	10	12	13	14	16	20			
Z_{13}	5	3	9	18	1	y	x	y^2	xy	y^3	x^2	xy^2	y^4	xy^3	y^5	xy^4	xy^5	
$x^3y+y^6+z^2$					0	3	5	6	8	9	10	11	12	14	15	17	20	
S_{11}	5	4	6	16	1	y	x	z	y^2	xy	yz	z^2	xy^2	y^2z	y^3z			
$x^2z+yz^2+y^4$					0	4	5	6	8	9	10	12	13	14	18			
W_{13}	4	3	8	16	1	y	x	y^2	xy	x^2	y^3	xy^2	x^2y	y^4	x^2y^2	y^5	y^6	
$x^4+xy^4+z^2$					0	3	4	6	7	8	9	10	11	12	14	15	18	
Q_{12}	6	5	3	15	1	z	y	x	z^2	yz	z^3	xy	yz^2	z^4	yz^3	yz^4		
$x^2z+y^3+z^5$					0	3	5	6	6	8	9	11	11	12	14	17		
S_{12}	4	3	5	13	1	y	x	z	y^2	xy	yz	y^3	xy^2	y^2z	y^4	y^5		
$x^2z+yz^2+xy^3$					0	3	4	5	6	7	8	9	10	11	12	15		
U_{12}	4	4	3	12	1	z	x	y	z^2	xz	yz	xy	xz^2	yz^2	xyz	xyz^2		
$x^3+y^3+z^4$					0	3	4	4	6	7	7	8	10	10	11	14		
U_{12}					1	z	x	y	z^2	xz	yz	y^2	xz^2	yz^2	y^2z	y^2z^2		
$x^2y+y^3+z^4$					0	3	4	4	6	7	7	8	10	10	11	14		

Lemma 2.8. *Assume symmetry g is quasihomogeneous. Then g is splitting if and only if $Jac(g) \notin \mathbb{R}$. In this case, the hyperbolic characters are $Jac(g)$ and its conjugate.*

Proof. According to [24], the rank 2 subspace in the cohomology $H^2(V_\star, \mathbb{C})$ dual to H_+ is spanned by the forms $\alpha = dx \wedge dy \wedge dz/dF_\star$ and $Hess(f)\,\alpha$. The two forms are eigenvectors of the automorphism $g^\star$ of $H^2(V_\star, \mathbb{C})$, with the eigenvalues $Jac(g)$ and its conjugate. □

Corollary 2.9. *Non-quasihomogeneous exceptional unimodal functions have no splitting symmetries.*

Indeed, a symmetry of such a function preserves the modular term $Hess(f)$. Hence both α and $Hess(f)\,\alpha$ are in the same character subspace in the cohomology.

Our main classificational result, on normal forms of splitting symmetries, is

Theorem 2.10. *Any invariant splitting symmetry g of a quasihomogenous exceptional unimodal singularity f falls into one of the following categories.*

- *a) The symmetry g of order $m > 2$ is a power of the Coxeter transformation C of function f.*
- *b) Each of the corank 2 singularities $E_{14}, Z_{13}, W_{13}, W_{12}$ admits symmetries g of order $m > 2$ which are powers of the Coxeter transformation composed with the involution $\iota_z(x, y, z) = (x, y, -z)$.*
- *c) Remaining symmetries are listed in Table 2.2.*

Table 2.2 lists the symmetries up to a choice of a different generator of the same cyclic group.

The involution ι_x used in Table 2.2 has been introduced in Section 2.1.3. According to Example 2.1, monomials to use in a g-miniversal deformation in case a) are exactly those of weights divisible by the order m of the symmetry g. It is clear that a similar choice in case b) coincides with that for

Table 2.2: Exceptional symmetries of Q_{12} and U_{12}

f	$g : x, y, z \mapsto$	$g =$	$\|g\|$	$g-$miniversal monomials	notation
$Q_{12}: \ x^2z + y^3 + z^5$	$\varepsilon_{10}^9 x, \omega y, \varepsilon_5 z$	$\iota_x C$	30	1	–
$\iota_x : (x, y, z) \mapsto$	$\varepsilon_{10}^7 x, y, \varepsilon_5^3 z$	$\iota_x C^3$	10	$1, y$	$Q_{12}\|\mathbb{Z}_{10}$
$\mapsto (-x, y, z)$	$-x, \omega^2 y, z$	$\iota_x C^5$	6	$1, z, z^2, z^3, z^4$	$Q_{12}\|\mathbb{Z}_6$
$U_{12}: \ x^3 + y^3 + z^4$	$\omega^2 x, y, iz$	σC	12	$1, y$	$U_{12}\|\mathbb{Z}_{12}$
$\sigma : (x, y, z) \mapsto$	$x, \omega y, -z$	σC^2	6	$1, x, z^2, xz^2$	$U_{12}\|\mathbb{Z}_6$
$\mapsto (\omega x, \omega^2 y, z)$	$\omega x, \omega^2 y, -iz$	σC^3	12	$1, xy$	$(U_{12}\|\mathbb{Z}_{12})'$
	$\omega^2 x, y, z$	σC^4	3	$1, z, y, z^2, yz, yz^2$	$U_{12}\|\mathbb{Z}_3$
$U_{12}: \ x^2y + y^3 + z^4$	$\varepsilon_6^5 x, \omega y, iz$	$\iota_x C$	12	1	–
$\iota_x : (x, y, z) \mapsto$	$\varepsilon_6 x, \omega^2 y, -z$	$\iota_x C^2$	6	$1, z^2$	$(U_{12}\|\mathbb{Z}_6)'$
$\mapsto (-x, y, z)$	$-x, y, -iz$	$\iota_x C^3$	4	$1, y, x^2, xz^2$	$(U_{12}\|\mathbb{Z}_4)'$
	$\varepsilon_6^5 x, \omega y, z$	$\iota_x C^4$	6	$1, z, z^2$	$(U_{12}\|\mathbb{Z}_3)'$

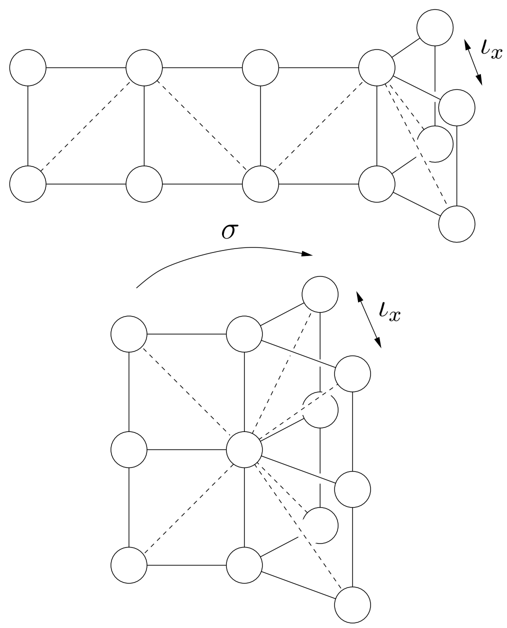


Figure 2.2: Symmetries of Dynkin diagrams of Q_{12} (above) and U_{12} (below).

the corresponding power of the Coxeter transformation. For the corank 2 functions not mentioned in part b), the symmetry $(x, y, z) \mapsto (x, y, -z)$ is $C^{N/2}$.

Theorem 2.10, in particular, states that, for any quasihomogeneous exceptional unimodal singularity f, we can make a quasihomogeneous coordinate change which diagonalises a splitting symmetry. In the case of U_{12}, there are two possible normal forms. This is similar to the two normal forms of the D_4 singularity.

The sign change in part b) of the Theorem is the $-id$ map on the vanishing homology. It does not affect the actual summands in the decomposition (2.2). It only affects the indexation, changing the signs of all characters.

The transformations ι_x and σ in Table 2.2 correspond to the order 2 and 3 symmetries of the Dynkin diagrams of the underlying singularities D_6 and D_4. The relevant symmetries of the Q_{12} and U_{12} Dynkin diagrams are shown in Figure 2.2 (the diagrams are constructed as those for the direct sums $D_6 \oplus A_2$ and $D_4 \oplus A_3$ of singularities, using the Gabrielov method [10]). Both ι_x and σ have real determinants, hence are able to split the subspace H_+ only in combination with a power of the Coxeter transformation which splits H_+ itself, that is, has order greater than 2.

2.2.2 Picard-Lefschetz operators

Since the character $Jac(g)$ has a special role, we will use a special notation χ' for it. In the direct sum

$$H^2(V_\star, \mathbb{C}) = \oplus_{\chi^m=1} H^\chi \,, \tag{2.3}$$

where the substitution $g^\star$ is multiplication by χ on H^χ, we have $\alpha = dx \wedge dy \wedge dz / dF_\star \in H^{\chi'}$. Each summand H^χ here is dual to the summand H_χ in (2.2).

We observe that the subspace $H^{\chi'}$ is the only summand in (2.3) that contains a holomorphic nowhere-vanishing 2-form. This helps us to find the eigenvalues of the Picard-Lefschetz operators acting on $H_{\chi'}$.

Proposition 2.11. *Consider the Picard-Lefschetz operator $h_{\chi'}$ on $H_{\chi'}$ corresponding to a g-orbit of critical points with a quasihomogeneous normal form $\psi(x', y', z')$. Choose the weights w_1', w_2', w_3' of the variables so that the weight of function ψ is 1. Then the eigenvalue of $h_{\chi'}$ is $\exp(2\pi i(w_1' + w_2' + w_3'))$.*

Proof. The restriction of the family F to a line germ transversal to Σ may be brought near any of the critical points to a local normal form $\psi(x', y', z') + \epsilon$. Locally, the cohomological operator $h^\star = \oplus h^\chi$ is induced by a loop in $\mathbb{C}_\epsilon$ going once around the origin in the positive direction. Its eigenvectors are the 2-forms $\omega_j = \alpha_j(x', y', z')\, dx' \wedge dy' \wedge dz'/d\psi$, where the α_j form a monomial basis of the local ring of function ψ. The transformation $h^\star$ is the substitution $x' := \exp(2\pi i w_1')\, x'$ etc. Hence its eigenvalue on ω_j is $\exp(2\pi i\, weight(\omega_j))$, where $weight(\omega_j) = weight(\alpha_j) + w_1' + w_2' + w_3'$.

The only eigenform ω_j that vanishes nowhere in a neighbourhood of our elementary critical point is the one in which α_j is a non-zero constant, that is, has weight 0. $\square$

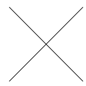
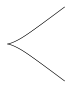
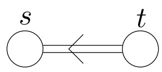
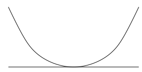
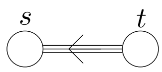
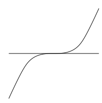
2.2.3 Dynkin diagrams

Our classification is encoded in the standard way using Dynkin diagrams. One interpretation of these diagrams gives a description of the structure of the discriminant $\Sigma \subset \Lambda$. Another interpretation we use says that a vertex corresponds to a generator of $\pi_1(\Lambda \setminus \Sigma)$ (equivalently M_χ), and the edges correspond to some of the relations. This part of the relations will be denoted by $\mathcal{B}$. Both interpretations are presented in Table 2.3. While we don't necessarily find a two dimensional section of the discriminant in each case, this interpretation still holds.

The exterior of each vertex is also labelled with the right equivalence class of the function germ corresponding to this component of the discriminant. The eigenvalue of the Picard-Lefschetz operator corresponding to the vertex coincides with the eigenvalue of the classical monodromy of this function germ, and the order of this is written in the interior of the vertex. This is the order of the generator of $M_{\chi'}$ corresponding to this vertex.

We also label the exterior of each vertex with the self-intersection number of the vanishing χ-cycle associated to this generator, and edges with the intersection numbers of neighbouring χ-cycles. The intersection number is zero if and only if there is no edge between vertices. See page 22, for example.

Table 2.3: Dynkin diagram relations on pairs of vertices

Singularity	Dynkin diagram	Local structure of Σ	$\mathcal{B}$-relations
$A_1 \times A_1$			$st = ts$
A_2			$sts = tst$
B_2			$(st)^2 = (ts)^2$
G_2			$(st)^3 = (ts)^3$

2.2.4 Equivariant splitting symmetries

Similar considerations can be made in the equivariant case by also considering constant multiples of the functions itself. We have the following classification theorem.

Theorem 2.12. *Any smoothable equivariant splitting symmetry g of a quasi-homogeneous exceptional unimodal singularity f falls into one of the following categories*

a) The symmetry g of order $m > 2$ is a fractional power of the Coxeter transformation C of function f.

b) Each of the corank 2 singularities $E_{14}, Z_{13}, W_{13}, W_{12}$ admits symmetries g of order $m > 2$ which are fractional powers of the Coxeter transformation composed with the involution $\iota_z(x, y, z) = (x, y, -z)$.

c) Remaining symmetries are powers of C composed with an invariant symmetry ι_x or σ (shown in Figure 2.2) which will be discussed in detail for Q_{12} on Page 102 and U_{12} on Page 117.

2.2.5 Dynkin-like diagrams

For equivariant symmetries, we recall a definition stated in [7]:

Definition 2.13. A group with ℓ generators acting on $\mathbb{C}^k$ is called *well presented* if $k = \ell$.

By convention, we draw the Dynkin diagram of a group with a 'curved edge' if and only if it is not well presented.

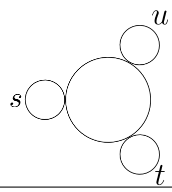
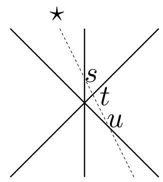
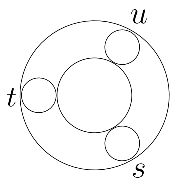
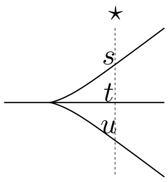
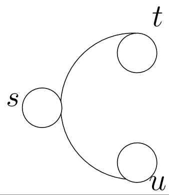
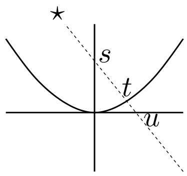
Definition 2.14. Such a diagram is called a *Dynkin-like diagram*, the 3-wise relations are called *braid-like*.

In [7] as well as in this thesis we find that the largest observed ℓ is $\ell = k+1$ (in the equivariant case only). Such groups may have generators which satisfy braid-like relations. All such relations necessary for this thesis can be found in Table 2.4 on Page 18, along with the Dynkin-like diagrams encoding these relations.

The diagrams given in Chapter 3 correspond to the same groups as those in Goryunov's papers, but in some cases may look different. Due to the amount of labelling frequently required, we introduce a new notation – that of the intersection diagram. If there is insufficient room to properly label a Dynkin diagram, it is followed by an intersection diagram containing all necessary information about intersection numbers. These diagrams have black vertices to distinguish them from Dynkin diagrams.

In Table 2.4 the dotted line is the generic line in the complement to the discriminant. Generators of the fundamental group of the complement are loops s, t, u labelled according to the diagrams in the table in the order in which the loops leave the base point $\star$. Loops leave $\star$ in an anticlockwise order and provide a distinguished basis.

Table 2.4: Dynkin-like diagram relations on triples of vertices

Dynkin diagram	Local structure of Σ	$\mathcal{B}$-relations
		$stu = tus = ust$
		$sutst = utsut, utsu = suts$
		$ustut = tustu, stu = tus$

2.2.6 Coincidence of weights

Definition 2.15. The graph obtained by removing all labels and orders of vertices of the Dynkin diagram is the *skeleton* of the Dynkin diagram.

As in Example 2.1, we take f to be a quasi-homogeneous function of degree N with respect to the weights $w_0, \ldots, w_n$ of the coordinates where $w_0, \ldots, w_n \in \mathbb{N}$, $\gcd(w_0, \ldots, w_n) = 1$. Choose the weights $v_1, \ldots, v_k$ of $\lambda_1, \ldots, \lambda_k$ in the unique way so that

$$F_g = f + \sum_{i=1}^{k} \lambda_i \varphi_i$$

is quasi-homogeneous, and assume they are arranged so that $v_i \leq v_{i+1}$ for all $i = 1, \ldots, k-1$.

Similarly, let G be a finite reflection group acting on $\mathbb{C}^k$ (as classified in [21]), basic invariants of which have degrees $m_1, \ldots, m_k$ also arranged so that $m_i \leq m_{i+1}$ for all $i = 1, \ldots, k-1$. An observation of our classification is the following.

Definition 2.16. A group with ℓ generators acting on a k dimensional vector space is said to have *corank* $\ell - k$.

Proposition 2.17. *Let $M_{\chi'}$ be a complex hyperbolic reflection group in our classification, and $v_1, \cdots, v_k$ the weights of the parameters in a corresponding g-miniversal deformation. Assume the ratio $(v_1 : \cdots : v_k)$ coincides with the ratio $(m_1 : \cdots : m_k)$ of degrees of basic invariants of a Shephard-Todd group G. If M_χ and G have equal coranks, then $M_{\chi'}$ and G have Dynkin diagrams with coinciding skeletons.*

Conversely, coincidence of the skeleton of a Dynkin diagram of $M_{\chi'}$ with that of some Shephard-Todd group G implies the equality of the weights ratio of $M_{\chi'}$ and the degrees ratio of G.

2.2.7 Discreteness of the monodromy group

We recall the following. Let H be an Hermitian form, and consider an algebraic number field E such there exists a totally real subfield F with $[E : F] = 2$. Use $\mathcal{O}_E$ to denote the ring of integers of E. Let $SU(H)$ denote the special unitary group defined by the form H, and $SU(\mathcal{O}_E, H) \subset SU(H)$ be the subgroup consisting of matrices with entries belonging to $\mathcal{O}_E$. There are a finite number of embeddings $\rho_i : F \to \mathbb{R}$. For each of these embeddings there is, up to complex conjugation, a unique compatible embedding $\tau_i : E \to \mathbb{C}$, from which we obtain a new Hermitian matrix ${}^{\tau_j}H$ by applying τ_j to the entries of H. The new associated group $SU({}^{\tau_j}H)$ is denoted by ${}^{\tau_j}SU(H)$.

Theorem 2.18 (Deligne, Mostow [9])**.** *The subgroup $SU(\mathcal{O}_E, H)$ is an arithmetic lattice in $SU(H)$ if and only if ${}^{\tau_j}SU(H)$ is compact for all non-trivial embeddings τ_j (up to complex conjugation).*

Corollary 2.19. *The projectivised versions of all groups appearing in the following classification are discrete subgroups of $SU(k-1, 1)$.*

Proof. For an exceptional unimodal singularity X, the non-symmetric monodromy group M is an infinite subgroup of the unitary group $U(\mu - 2, 2)$.

After taking the quotient of the Milnor fibre by the symmetry group generated by g the group is the direct sum of groups, each of dimension $k_\chi \leq k$ (by observation).

$$M = \bigoplus_{\chi:\chi^m=1} M_\chi.$$

Due to Lemma 2.8, we assume $\chi' \notin \mathbb{R}$. Let χ' (and therefore its conjugate) be such that $M_{\chi'} \subset U(k-1,1)$. Since the sum of signatures of the character subspaces must equal the signature of their direct sum, for all $\chi \neq \chi'$ (up to complex conjugation) we have $M_\chi \subset U(k_\chi)$, which is compact. After projectivisation the compactness argument also holds, and moreover entries in the matrices belong to the cyclotomic ring $\mathbb{Z}\langle\chi\rangle$. The field $\mathbb{Q}\langle\chi\rangle$ is an imaginary quadratic extension of the totally real subfield $\mathbb{Q}\langle\chi+\overline{\chi}\rangle$ since the minimal polynomial of χ in $\mathbb{Q}\langle\chi+\overline{\chi}\rangle$ is

$$x^2 - (\chi+\overline{\chi})x + 1.$$

The ring of integers is $\mathbb{Z}\langle\chi\rangle \subset \mathbb{Q}\langle\chi\rangle$, and so all necessary criteria is satisfied.

$\square$

Chapter 3

Known results

This chapter gives all results coming from other papers which are used in the classification in Chapter 4.

3.1 Lifts of simple boundary singularities

We recall all results from [11, 12, 13] relevant to this thesis.

Example 3.1. The boundary singularity A_2 has normal form $x^3 + y_0 + z^2$, boundary given by $\{y_0 = 0\}$. We take an m-fold covering of $\mathbb{C}^3$ ramified along the boundary by setting $y_0 = y^m$. This is similar to taking a singularity $X \ni x^3 + y^m + z^2$ of codimension $2(m-1)$, and considering $X|\mathbb{Z}_m$ with the symmetry $g : (x, y, z) \mapsto (x, \varepsilon_m y, z)$. We consider the picture without the variable z.

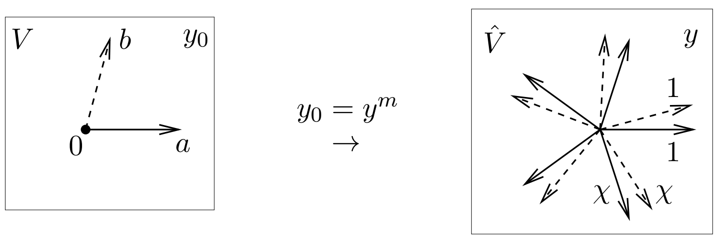


Figure 3.1: Taking m-fold covers of semi-cycles a and b.

This singularity has two χ-cycles $\hat{a}$ and $\hat{b}$, lifted from semi-cycles a and b in the homology of the Milnor fibre of A_2 in the standard way as described in Section 2.1.4. We use $\hat{V}$ to denote the boundary cover of the Milnor fibre V by the substitution $y+0=y^m$. Figure 3.2 shows that $\left\langle (1-\chi)\hat{a}, \hat{b} \right\rangle = m$, where the solid paths are obtained by multiplying the solids paths corresponding to $\hat{a}$ in Figure 3.1. The suspension of cycles into an extra dimension by adding the variable z^2 implies self-intersection number $-m$ for both $\hat{a}, \hat{b}$. The intersection form is Hermitian for an odd number of variables, which also gives for 3 variables

$$\left\langle \hat{a}, \hat{b} \right\rangle = \frac{m}{1-\chi}.$$

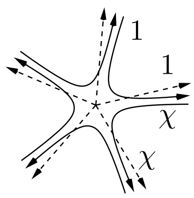
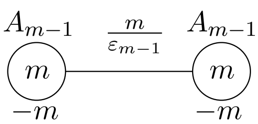


Figure 3.2: Proof that $\left\langle (1-\chi)\hat{a}, \hat{b} \right\rangle = m$.

The intersection matrix is therefore

$$\begin{pmatrix} -m & -\frac{m}{\chi-1} \\ -\frac{m}{\overline{\chi}-1} & -m \end{pmatrix},$$

where $\chi^m = 1, \chi \neq 1$. The function germ representative corresponding to the only component of the discriminant is of right equivalence class A_{m-1}. The eigenvalue coming from the classical monodromy of this is of order dividing m. We want our groups to be hyperbolic, giving the constraint $|\chi - 1| < 1$, so we must take $\chi = \varepsilon_m$ or $\chi = \overline{\varepsilon_m}$ if $6 < m < 13$. So we get the following diagram for $A_2^{(m)}$.



$A_2^{(m)}$

Pairs of connected vertices in the Dynkin diagram for a singularity of type $A_\mu^{(m)}, D_\mu^{(m)}, E_\mu^{(m)}$ have the same local structure as $A_2^{(m)}$, and the labelling for each pair is the same as in the diagram above.

An ambiguity occurs when we label cycles with powers of χ (e.g. see Figure 3.1) since we could start this labelling process on any cycle, and when we choose orientations. For this reason, the intersection number of the two cycles is defined only up to multiplication by a power of ε_m or by ± 1. It is shown in [11] that adding a non-degenerate quadratic form in an even number of variables to a function affects the intersection numbers only perhaps by further multiplication by -1. Moreover, if the Dynkin diagram is a tree this ambiguity affects every edge and we have freedom to choose the most convenient intersection number.

Oriented edges of Dynkin diagrams indicate the order in which we take intersection numbers:

$$a \xrightarrow{U} b,$$

means $\langle a, b\rangle = U$. However the reorientation

$$a \xleftarrow{\overline{U}} b$$

may easily be brought to the form

$$a \xleftarrow{U} b$$

by the ambiguities mentioned in the previous paragraph. So in the case when the Dynkin diagram is a tree (specifically this example), we do not assign orientations to the edges of the diagram.

Example 3.2. Starting with the boundary singularity $B_2 \ni y_0^2$ ($\{y_0 = 0\}$ the boundary) and setting $y_0 = y^m$, we obtain the singularity $B_2^{(2,m)}$. To construct its diagram we use an explicit example as in [11]. Consider the covering $y_0 = y^m$ of a deformation $f(y_0) = y_0^2 - 4y_0 + 3$ of the one variable boundary singularity B_2. Set $\hat{f}(y) = f(y^m)$. Take $\hat{f} = 0$ as the Milnor fibre $\widehat{V}$ covering $f = 0$ (see figure 3.3). Join $0 \in \mathbb{C}$ by the straight paths with critical values $\hat{f}(0) = 3$ and $\hat{f}(2) = 1$ of $\hat{f}$. Take the linear combination of the points on the inner circle in Figure 3.3 for a $\chi-$cycle $e_1 \in H_\chi \subset \overline{H}_0(\widehat{V})$ (the reduced 0-degree homology) vanishing on the level $\hat{f} = 3$, and the difference between the linear combinations on the outer and inner circles for a $\chi-$cycle e_2 vanishing on $\hat{f} = -1$. The intersection matrix $(\langle e_i, e_j\rangle)$ is

$$\begin{pmatrix} m & -m \\ -m & 2m \end{pmatrix}.$$

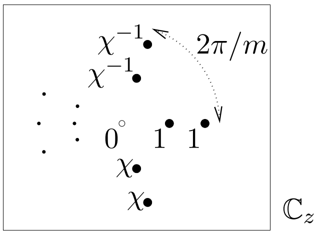


Figure 3.3: Cycles for B_2

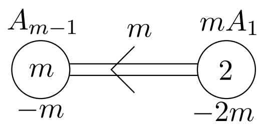
The Picard-Lefschetz operator h_1 rotates the points on the inner circle anti-clockwise by $2\pi/m$ which means

$$\begin{aligned} h_1(e_1) &= \chi e_1, \\ h_1(e_2) &= e_2 + (1-\chi)e_1 \\ &= e_1 - (1-\chi)\langle e_2, e_1\rangle e_1/m \\ &= e_1 + (\chi - 1)\frac{\langle e_2, e_1\rangle}{\langle e_1, e_1\rangle} e_1. \end{aligned}$$

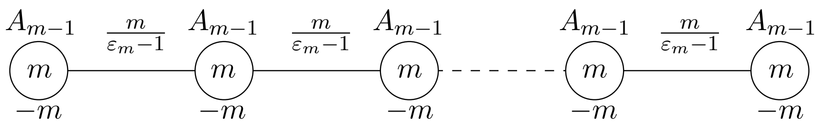
The operator h_2 swaps points on the same ray from the origin:

$$\begin{aligned} h_2(e_1) &= e_1 + e_2 \\ &= e_1 - \langle e_1, e_2\rangle e_2/m \\ &= e_1 + (e^{2\pi i/2} - 1)\frac{\langle e_1, e_2\rangle}{\langle e_2, e_2\rangle} e_2, \\ h_2(e_2) &= -e_2. \end{aligned}$$

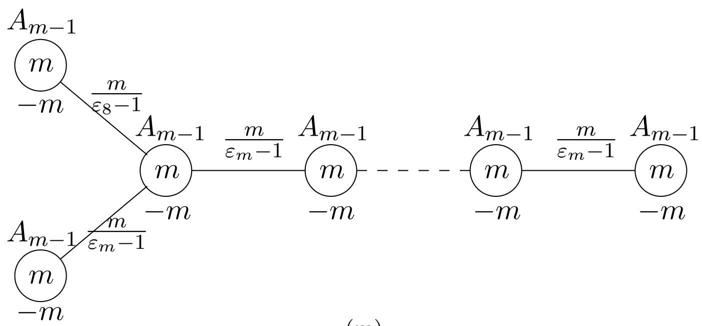
As seen in Figure 3.3, we obtain the Dynkin diagram for $B_2^{(m,2)}$ (which is sometimes written as $B_2^{(m)}$).



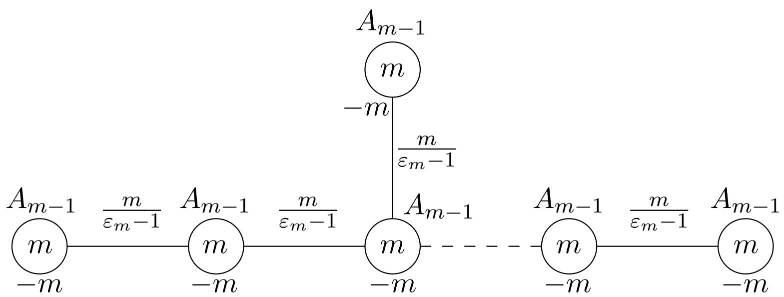
Together with the result for $A_2^{(m)}$, we may now construct Dynkin diagrams for similar lifts of all simple boundary singularities. In each case there are μ vertices. Dashed edges in the following diagrams indicate the diagram should be continued by a chain of vertices.



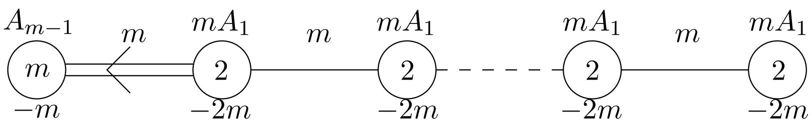
$A_\mu^{(m)}$



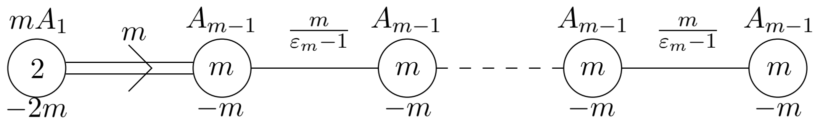
$D_\mu^{(m)}$



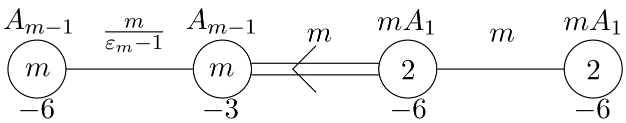
$E_\mu^{(m)}$



$B_\mu^{(m)}$



$C_\mu^{(m)}$



$F_4^{(m)}$

From now on, we will choose the special character χ', the eigenvalue of $g^\star$ on $\frac{dxdydz}{df}$, details of which are given in Section 2.2.1, so that the monodromy group in our case is complex hyperbolic.

3.2 Particular intersection numbers

In [12], a series of invariant singularities

$$\begin{aligned} D_{m+1}|\mathbb{Z}_{2m} &\ni x^2y + y^m + z^2; \\ g(x, y, z) &= (\overline{\varepsilon_{2m}}x, \varepsilon_m y, z) \end{aligned}$$

was introduced. The self-intersection number of the χ-cycle e corresponding to this singularity is $\langle e, e\rangle = m(-2 + \chi + \overline{\chi})$, $\chi^m = -1$. In particular, we will use the following with $\chi = \varepsilon_{2m}$:

- For $D_4|\mathbb{Z}_6$, $\langle e, e\rangle = 3(-2 + \varepsilon_6 + \overline{\varepsilon_6}) = -3$,

- For $D_5|\mathbb{Z}_8$, $\langle e, e\rangle = 4(-2+\varepsilon_8+\overline{\varepsilon_8}) = 4\sqrt{2}-8$,
- For $D_6|\mathbb{Z}_{10}$, $\langle e, e\rangle = 5(-2+\varepsilon_{10}+\overline{\varepsilon_{10}}) = \frac{5}{2}(\sqrt{5}-3)$.

We also need the self-intersection number of a $\chi-$cycle e corresponding to a singularity of type $E_6|\mathbb{Z}_{12}$, $\chi = -i\omega = \varepsilon_{12}$. According to [19], this is

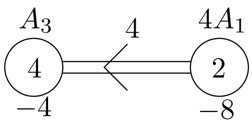
$$\langle e, e\rangle = -2\cdot 3\cdot 4\frac{1-(-1)\cdot\omega\cdot i}{(1-(-1))(1-\omega)(1-i)} = 2\sqrt{3}-6.$$

3.3 Symmetries of simple singularities

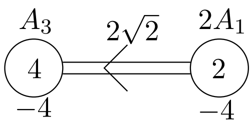
Full details of the following singularities with invariant symmetry are given in [12], the singularity with equivariant symmetry in [13].

$\underline{B_2^{(4)}}$

For $f = x^8+yz \in A_7$ with the invariant symmetry $g(x,y,z) = (ix,y,z)$, g-versal monomials are $1, x^4$ and the Dynkin diagram is given.

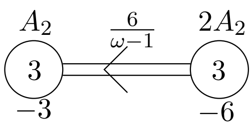


A group with the same notation occurs as a symmetry of D_5. Namely, for $f = x^2y+y^4+z^2$ with the invariant symmetry $g(x,y,z) = (-ix,y,-z)$, g-versal monomials are $1, y^2$, and the Dynkin diagram is given.



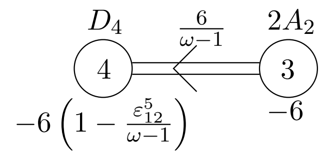
$\underline{B_2^{(3,3)}}$

For $f = x^3+y^4+z^2 \in E_6$ with the invariant symmetry $g(x,y,z) = (\omega x,-y,z)$, g-versal monomials are $1, y^2$ and the Dynkin diagram is given.



$\underline{B_2^{(4,3)}}$

For $f = x^3 + y^5 + z^2 \in E_8$ with equivariant symmetry $g(x, y, z; f) = (-\omega x, -y, iz; -f)$, g-versal monomials are y, y^3 and the Dynkin diagram is given.



3.4 Symmetries of P_8, X_9 and J_{10}

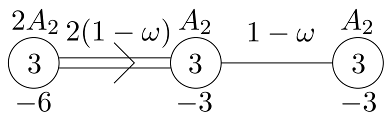
Here we display some of known Dynkin diagrams of groups coming from symmetries of P_8, X_9 and J_{10} which can be found in [14, 15, 16] respectively. We fix the character such that these groups are complex crystallographic reflection groups, and extending these groups in our classification yields complex hyperbolic reflection groups. We omit some of those groups which are isomorphic to ones already mentioned in this chapter.

3.4.1 From P_8

The following results come from [14].

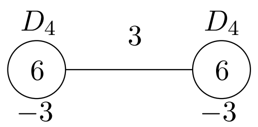
$\underline{C_3^{(3,3)}}$

For $f = x^3 + y^3 + yz^2 \in P_8$ with the invariant symmetry $g(x, y, z) = (\omega x, y, -z)$, g-versal monomials are $1, y, y^2$ and the Dynkin diagram is given.



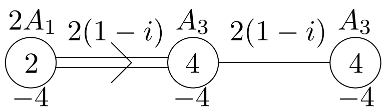
$\underline{(P_8|\mathbb{Z}_6)'}$

For $f = x^3 + y^3 + yz^2 \in P_8$ with the invariant symmetry $g(x, y, z) = (x, \overline{w}y, -\overline{w}z)$, g-versal monomials are $1, x$ and the Dynkin diagram is given.



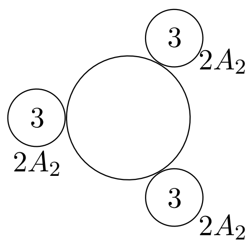
<u>$C_2^{(2,4)}$</u>

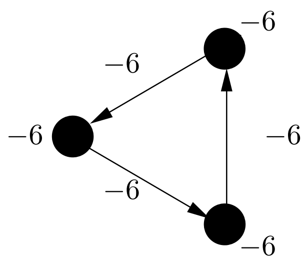
For $f = x^2z + xy^2 + z^3 \in P_8$ with the invariant symmetry $g(x, y, z) = (-x, -iy, z)$, g-versal monomials are $1, z, z^2$ and the Dynkin diagram is given.



<u>$P_8/\mathbb{Z}_6$</u>

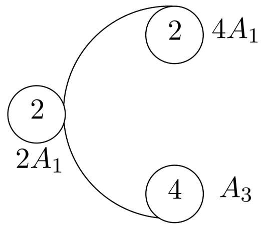
For $f = x^3 + y^3 + z^3 \in P_8$ with the equivariant symmetry $g(x, y, z; f) = (-\omega x, -y, -z; -1)$, g-versal monomials are y, z and the Dynkin-like and intersection diagrams are given.

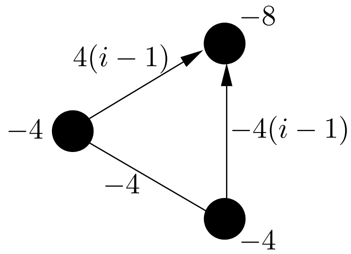




<u>$P_8/\mathbb{Z}_4$</u>

For $f = x^2z + xy^2 + z^3 \in P_8$ with the equivariant symmetry $g(x, y, z; f) = (ix, -y, -iz; if)$, g-versal monomials are x, yz and the Dynkin-like diagram is given.



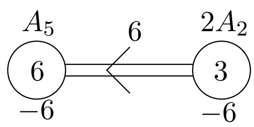


3.4.2 From X_9

The following results come from [15].

$B_2^{(6,3)}$

For $f = x^4 + xy^3 + z^2 \in X_9$ with the invariant symmetry $g(x, y, z) = (-x, -\omega y, z)$, g-versal monomials are $1, x^2$ and the Dynkin diagram is given.

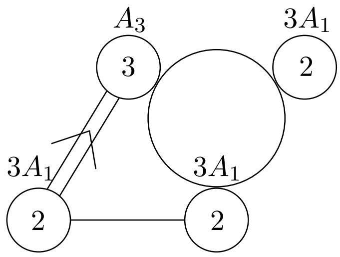


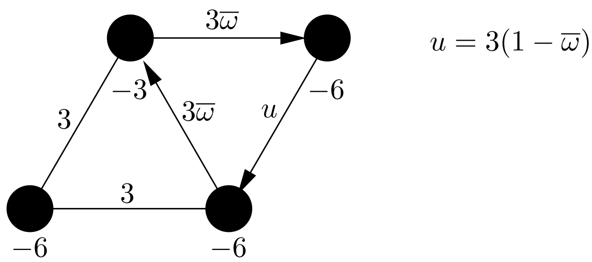
$X_9/\mathbb{Z}_6$

The paper [15] does not use Dynkin-like diagrams, so we perform our own calculations. For $f = x^4 + xy^3 + z^2 \in X_9$ with the equivariant symmetry $g(x, y, z; f) = (\omega x, \overline{\omega} y, \overline{\omega} z; \omega f)$. We have

$$\begin{aligned} F_g &= x^4 + xy^3 + z^2 + \gamma x^2 y + \beta y^2 + \alpha x \\ F_{g,x} &= 4x^3 + y^3 + 2\gamma xy + \alpha \\ F_{g,y} &= 3xy^2 + \gamma x^2 + 2\beta y \end{aligned}$$

The first two components of the discriminant are $\Sigma_1 = \{\alpha = 0\}$ and $\Sigma_2 = \{27\alpha^2+\gamma^3\alpha-16\beta^3-36\gamma\beta\alpha+8\gamma^2\beta^2-\gamma^4\beta\}$. The union $\Sigma_1 \cup \Sigma_2$ is a standard B_3 type discriminant. The final component $\Sigma_3 = \{\beta = 0\}$ gives a line producing a triple point in a generic section. All other intersections are transversal. The Dynkin-like and intersection diagrams follow.





3.4.3 From J_{10}

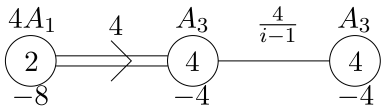
The following results come from [16].

$J_{10}|\mathbb{Z}_3$

For $f = x^3 + y^6 + z^2 \in J_{10}$ with the invariant symmetry $g(x, y, z) = (\omega x, \omega y, z)$, g-versal monomials are $1, y^3, xy^2$ and the Dynkin diagram is given.

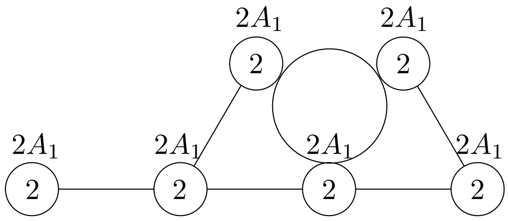
$\underline{C_3^{(4)}}$

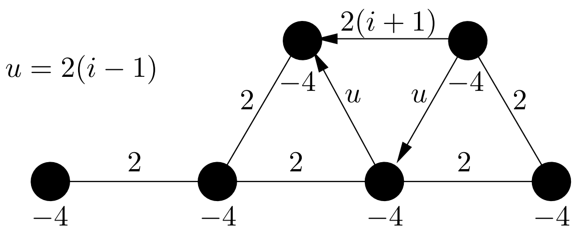
For $f = x^3+xy^4+z^2 \in J_{10}$ with the invariant symmetry $g(x, y, z) = (x, iy, z)$, g-versal monomials are $1, x, y^4$ and the Dynkin diagram is given.



$\underline{J_{10}/\mathbb{Z}_4}$

For $f = x^3 + xy^4 + z^2 \in J_{10}$ with the equivariant symmetry $g(x, y, z; f) = (-x, -y, iz; -f)$, g-versal monomials are y, y^3, y^5, x, xy^2 and the Dynkin-like diagram is given.





Chapter 4

The Classification

In this chapter we list all splitting symmetries of the 14 exceptional unimodal singularities given in Table 2.1. Each symmetry is presented with an accompanying Dynkin (or Dynkin-like) diagram encoding all necessary information to describe the monodromy subgroup with hyperbolic signature. When considering a singularity of right equivalence class X with respect to a symmetry g of order m, we write $X|\mathbb{Z}_m$ in the invariant case, $X/\mathbb{Z}_m$ in the equivariant case, or one of these with dashes to distinguish between symmetries of the same order with different g-versal deformations. Some lifts of simple singularities are already well known in the literature, for others we may use the following shortcuts in calculation.

1. If $X|\mathbb{Z}_m \to Y|\mathbb{Z}_m$ with the same symmetry g then the Dynkin diagram of $Y|\mathbb{Z}_m$ is a sub-diagram of some Dynkin diagram of $X|\mathbb{Z}_m$. In the cases Y is one of the parabolic singularities P_8, X_9, J_{10}, the details can be found in [14, 15, 16] respectively; if Y is a simple singularity the details can be found in [11, 12, 13]. All relevant information from these papers is reproduced in Chapter 3 and specific parts will be referenced as we proceed.

2. The following may occur. Let X admit an order m symmetry g where $g(z_i) = z_i$. If X also admits an order $2m$ symmetry g', defined by $g'(z_i) = -z_i$ and $g'(z_j) = g(z_j)$ such that $F_g \not\equiv F_{g'}$, then the Dynkin diagram for $X|\mathbb{Z}_{2m}$ is a folding of the Dynkin diagram for $X|\mathbb{Z}_m$ in the

usual sense. For example, take $E_6 \ni f = x^4 + y^3 + z^2$ with $g = id$, a trivial symmetry, and $g'(x, y, z) = (-x, y, z)$ of order 2. Then the normal form $E_6|\mathbb{Z}_2$, i.e. after the substitution $x^2 = x_0$, is $x_0^2 + y^3 + z^2 \in F_4$. Therefore $E_6|\mathbb{Z}_2 = F_4$, and this is demonstrated visually by folding the Dynkin diagram. See Figure 4.1.

3. It is not necessary to calculate unknown intersections geometrically. Say our Dynkin diagram for $X|\mathbb{Z}_m$ is fully labelled except for an intersection number U of two particular cycles. We may write the matrices of our Picard-Lefschetz operators in terms of U, and observe that these should satisfy certain braiding relations. These give a condition on $|U|^2$. We further observe that $U \in \mathbb{Z}\langle\chi'\rangle$. We write a general element of this ring and impose the value of $|U|^2$. This usually gives a system of diophantine equations with $lcm(2, m)$ solutions which we calculate by brute force using the MAPLE code given in Section 6.1.

4. In the invariant case, the Milnor number μ of X is preserved as the sum of multiplicities of singularities corresponding to intersection points of the discriminant with the generic line. We will refer to this as $\Sigma\mu_i = \mu$ in what follows. A similar equivariant constraint will be discussed later.

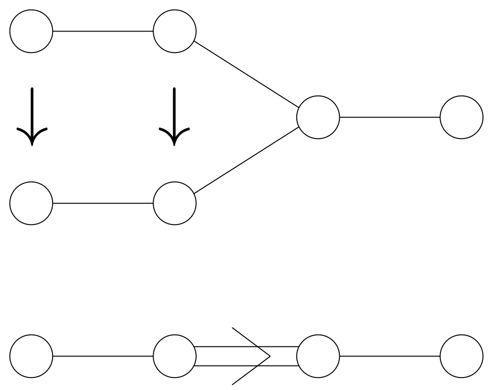


Figure 4.1: Folding the Dynkin diagram of E_6 to that of F_4.

Symmetries are given only up to coinciding symmetric versal deformations. We do not include symmetries of order 2 since by Lemma 2.8 we require the eigenvalue of g not to be real.

In each case, we assume the symmetry g has the standard form $g(x, y, z) = (ax, by, cz)$, where a, b, c are to be found. Solving $f \circ g(x, y, z) = f$ means solving a system of three equations in a, b, c, written

$$a^{\alpha_i} b^{\beta_i} c^{\gamma_i} = 1, \quad i = 1, 2, 3.$$

Here $\alpha_1, \beta_1, \gamma_1$ are exponents in the monomial summand $x^{\alpha_1} y^{\beta_1} z^{\gamma_1}$ of the singularity. The absolute value of the determinant of the matrix of exponents

$$\Delta = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

gives the number of solutions to this system. If $\Delta = N$, the order of the Coxeter element C, then all solutions, and therefore all symmetries, must be powers of C. If the number of solutions to this system is equal to $2N$ and the function germ $f \in X$ is stably equivalent to a function of two variables, then by observation we see that symmetries are powers of C combined with powers of the involution $\iota_z(x, y, z) = (x, y, -z)$. Since z does not appear in the local ring of such a function germ and we are classifying only up to coinciding deformations we ignore such symmetries, only considering powers of C.

For each group appearing in this classification the generators have been entered into a MAPLE worksheet in order to check that the calculations have been done correctly. Indeed, is has been tested that the generators do in fact satisfy all relations they are claimed to in this thesis.

4.1 Summary of interesting results

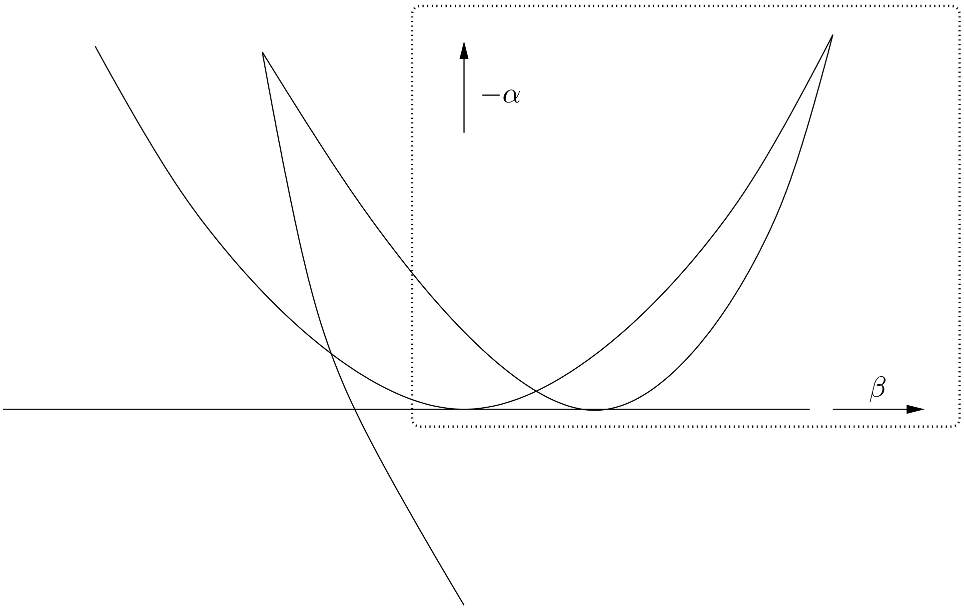
In the sections that follow the invariant and equivariant symmetries are exhausted in detail for each exceptional unimodal singularity. Many of the groups appearing have either appeared in other papers or are calculated through some elementary methods. Since there are many results, we are isolating some of the most interesting examples and presenting an exposition of these examples in this section. Displayed next to each singularity is the page number to find full details, and the ratio of weights of deformation parameters. Unless explicitly stated otherwise, the ratios found in this section have not appeared as weights of basic invariants in the Shepard-Todd classification [21] for groups of the same corank, nor have the skeletons of the Dynkin(-like) diagrams appeared as linear complex reflection groups. See Proposition 2.17. Readers are reminded that groups of corank 0 have Dynkin diagrams with only straight edges, groups of corank 1 have Dynkin-like diagrams with some curved edges, and groups of higher corank do not appear in this thesis.

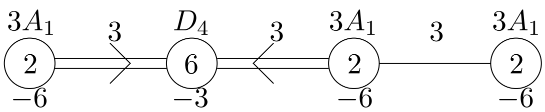
<u>$E_{13}|\mathbb{Z}_6$, $(1:2:3:5)$, page 47</u>

This singularity is adjacent to a known singularity

$$E_{13}|\mathbb{Z}_6 \to J_{10}|\mathbb{Z}_3,$$

details of which can be found in Section 3.4.3. The figure below display a generic 2 dimensional section of the discriminant of $E_{13}|\mathbb{Z}_6$, and within this a generic 2 dimensional section of the discriminant of $J_{10}|\mathbb{Z}_3$ is boxed.



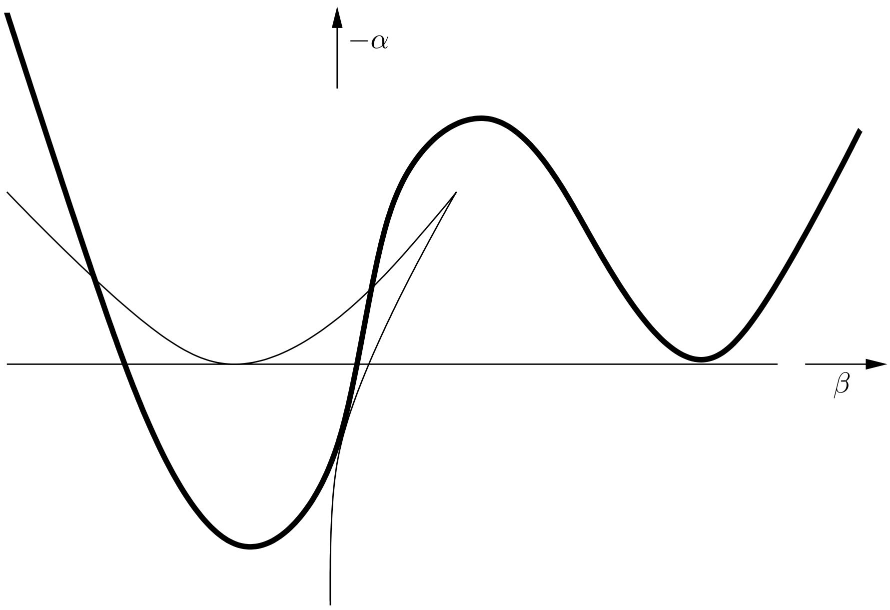


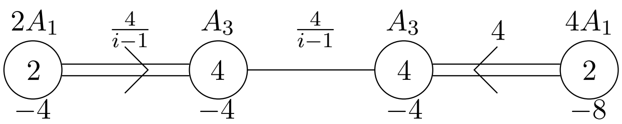
$(U_{12}|\mathbb{Z}_4)'$, $(1:2:4:6)$, page 74

We have

$$(U_{12}|\mathbb{Z}_4)' \to J_{10}|\mathbb{Z}_4 \cong C_3^{(4)}.$$

See Section 3.4.3 for details. In the generic 2 dimensional section of $(U_{12}|\mathbb{Z}_4)'$ that follows, one component is displayed in bold to distinguish it from the others. Removing this component, we see a generic 2 dimensional section of the $C_3^{(4)}$ discriminant.



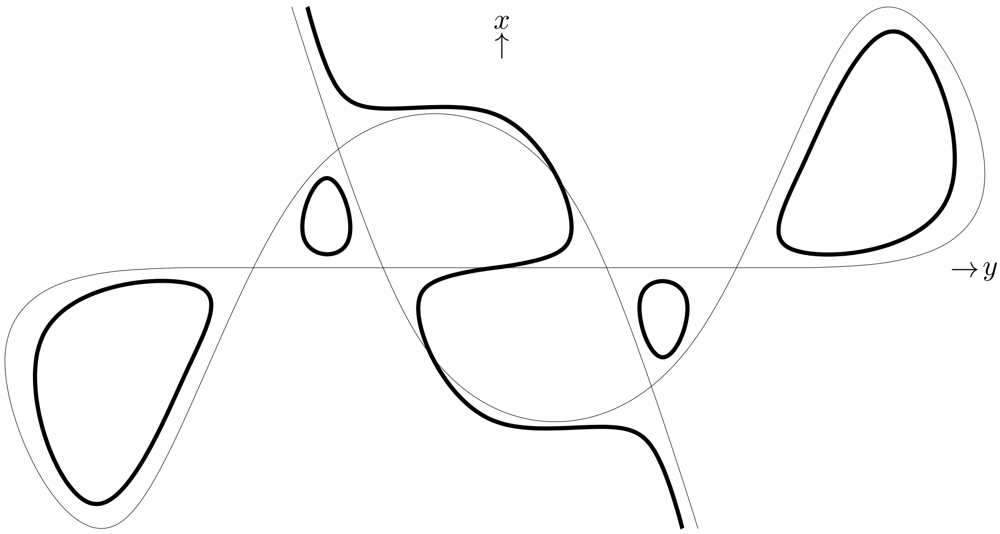


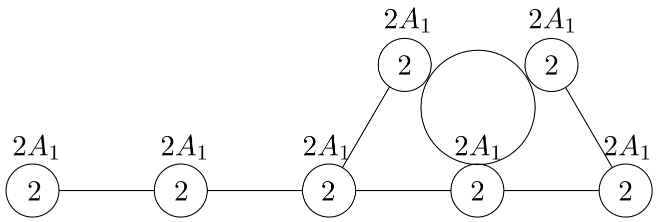
$E_{12}/\mathbb{Z}_4$, $(1:3:4:6:7:9)$, page 82

We have

$$E_{12}/\mathbb{Z}_4 \to J_{10}/\mathbb{Z}_4.$$

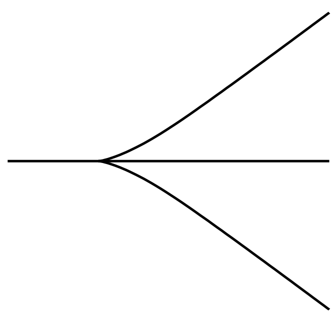
See Section 3.4.3. A sabirification of the deformation F_g of $E_{14}/\mathbb{Z}_4$ exists with the zero level set shown below (the bold curve is a smoothening).

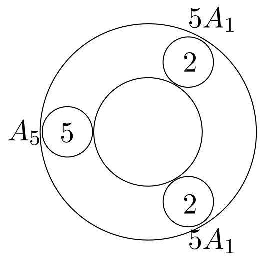




<u>$Q_{10}/\mathbb{Z}_5$, $(2:3)$, page 85</u>

The ratio $(2:3)$ is the same as that of an A_2 singularity, but in the $Q_{10}/\mathbb{Z}_5$ case the monodromy group M_χ has corank 1. The discriminant is isomorphic to that of the A_2 discriminant with a line through the origin tangent to the cusp.





The generator in the bottom right of the Dynkin-like diagram is h_1, and travelling clockwise generators are h_1, h_2, h_3. Generators satisfy the relations

$$\begin{aligned} h_1h_3h_2h_1h_2 - h_2h_1h_3h_2h_1 &= 0 \\ h_3h_2h_1h_3 - h_1h_3h_2h_1 &= 0. \end{aligned}$$

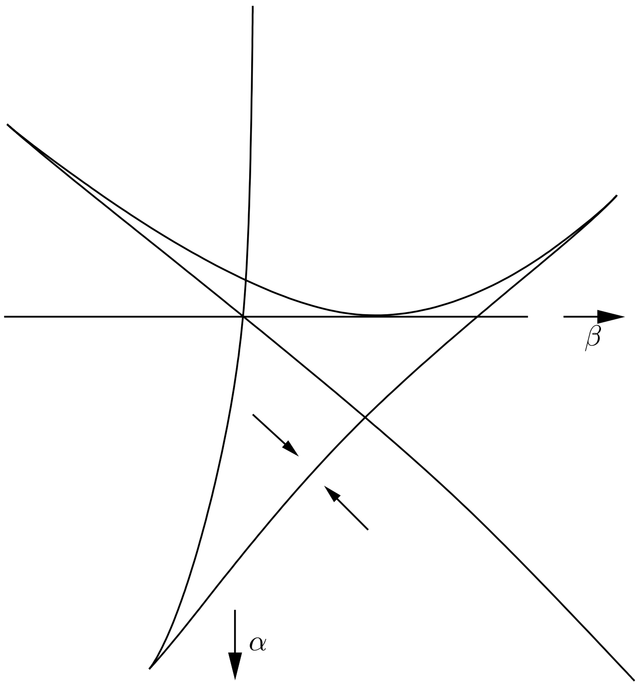
These relations and indeed the skeleton of this diagram have been seen in [7] for the Shephard-Todd group G_{13}. The weights of basic invariants of this group are also in the ratio $(2:3)$.

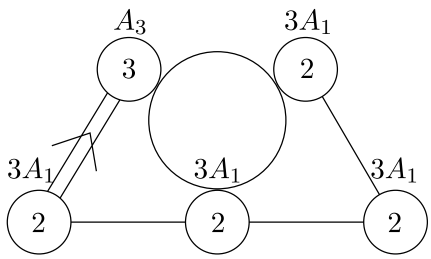
$W_{12}/\mathbb{Z}_6$, $(1:2:4:5)$, page 97

We have

$$W_{12}/\mathbb{Z}_6 \to X_9/\mathbb{Z}_6.$$

Details can be found in Section 3.4.2. A generic section of the discriminant of $X_9/\mathbb{Z}_6$ has three components: one component isomorphic to that of a B_3 discriminant, two lines intersecting at the origin. It can be seen in the generic section of the discriminant of $W_{12}/\mathbb{Z}_6$ by removing the marked edge connected two cusps.





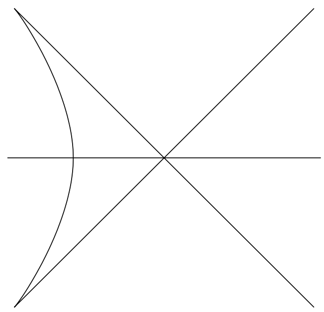
<u>$Q_{12}/\mathbb{Z}_6$, $(2:3:4)$, page 107</u>

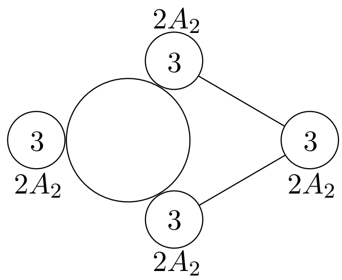
We have

$$Q_{12}/\mathbb{Z}_6 \to P_8/\mathbb{Z}_6,$$

details of which can be found in Section 3.4.1. The discriminant in the P_8 case is just three intersecting lines, which can be seen near the triple point in the generic section of the discriminant of $Q_{12}/\mathbb{Z}_6$.

This ratio $(2:3:4)$ is the same as the ratio associated with a standard A_3 singularity, the discriminant of which is a swallowtail. In our case the monodromy group has corank 1 and the swallowtail is intersected with a plane. A generic two dimensional section is given. This ratio also coincides with that of the Shephard-Todd group $G(6,2,3)$, the Dynkin-like diagram of which has the same skeleton. Note that a subdiagram of this is the standard $A_3^{(3)}$ diagram.



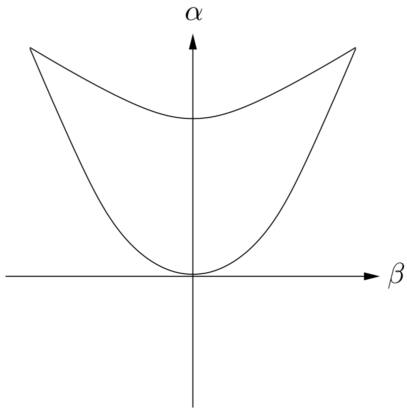


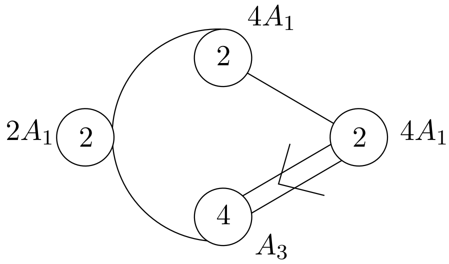
$Q_{12}/\mathbb{Z}_4$, $(2:3:6)$, page 109

We have

$$Q_{12}/\mathbb{Z}_4 \to P_8/\mathbb{Z}_4,$$

details of which can be found in Section 3.4.1. A section of the $Q_{12}/\mathbb{Z}_4$ discriminant is given, but this section is not generic. This is explained fully in the classification.





In the diagram, the generator on the far right is h_4. Travelling around clockwise from here, the generators are h_4, h_1, h_3, h_2. The full set of braiding relations associated to the skeleton of the Dynkin-like diagram is

$$\begin{aligned}
\gamma_1\gamma_3\gamma_2\gamma_1\gamma_2 &= \gamma_2\gamma_1\gamma_3\gamma_2\gamma_1 \\
\gamma_2\gamma_1\gamma_3 &= \gamma_3\gamma_2\gamma_1 \\
\gamma_4\gamma_3 &= \gamma_3\gamma_4 \\
\gamma_2\gamma_4\gamma_2 &= \gamma_4\gamma_2\gamma_4 \\
(\gamma_1\gamma_4)^2 &= (\gamma_4\gamma_1)^2.
\end{aligned}$$

4.2 Classification of Invariant Symmetries

4.2.1 $E_{12} \ni x^3 + y^7 + z^2$

The Coxeter element is $C(x,y,z) = (\omega x, \varepsilon_7 y, -z)$. We wish to find all symmetries $g(x,y,z) = (ax, by, cz)$. That is, find all solutions to the system of equations

$$\begin{aligned} a^3 &= 1 \\ b^7 &= 1 \\ c^2 &= 1. \end{aligned}$$

The determinant of the matrix of exponents is

$$\Delta = \begin{vmatrix} 3 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 42.$$

Since $\Delta = N$, all symmetries are powers of the Coxeter element.

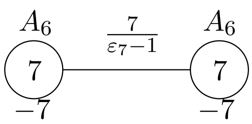
f	$g =$	$\lvert g\rvert$	versal monomials	notation
$x^3 + y^7 + z^2 \in E_{12}$	C, C^2	$42, 21$	1	-
	C^3, C^6	$14, 7$	$1, x$	$E_{12}\vert\mathbb{Z}_7$
	C^7, C^{14}	$6, 3$	$1, y, y^2, y^3, y^4, y^5$	$E_{12}\vert\mathbb{Z}_3$

These singularities have already been seen in Section 3.1:

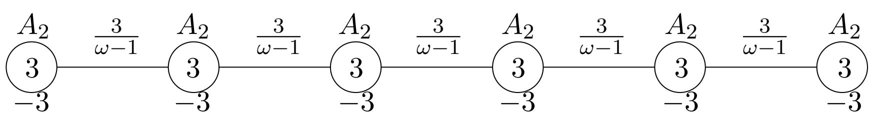
$$\begin{aligned} E_{12}|\mathbb{Z}_7 &\cong A_2^{(7)} \\ E_{12}|\mathbb{Z}_3 &\cong A_6^{(3)}, \end{aligned}$$

immediately giving us the Dynkin diagrams below.

$E_{12}|\mathbb{Z}_7$



$E_{12}|\mathbb{Z}_3$



4.2.2 $Z_{11} \ni x^3y + y^5 + z^2$

The Coxeter element is $C(x,y,z) = (\varepsilon_{15}^4 x, \varepsilon_5 y, -z)$. A general invariant symmetry is $g(x,y,z) = (ax, by, cz)$, where a, b, c satisfy

$$\begin{aligned} a^3b &= 1 \\ b^5 &= 1 \\ c^2 &= 1. \end{aligned}$$

The determinant of the matrix of exponents is

$$\Delta = \begin{vmatrix} 3 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 30.$$

Since $\Delta = N$, all invariant symmetries are powers of C.

f	$g =$	$\lvert g \rvert$	versal monomials	notation
$x^3y + y^5 + z^2 \in Z_{11}$	C, C^2	$30, 15$	1	-
	C^3, C^6	$10, 5$	$1, xy^2$	$Z_{11}\lvert\mathbb{Z}_{10}$
	C^5, C^{10}	$6, 3$	$1, y, y^2, y^3, y^4$	$Z_{11}\lvert\mathbb{Z}_6$

$Z_{11}|\mathbb{Z}_{10}$

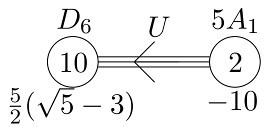
For $Z_{11}|\mathbb{Z}_{10}$, we have

$$\begin{aligned} F_g &= x^3y + y^5 + z^2 + \beta xy^2 + \alpha \\ F_{g,x} &= 3x^2y + \beta y^2 \\ F_{g,y} &= x^3 + 5y^4 + 2\beta xy. \end{aligned}$$

For $\alpha = 0$, F_g has a normal form

$$F_g|_{\alpha=0} \sim xy^2 + x^5 + z^2,$$

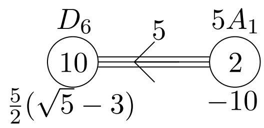
which has a D_6 singularity at the origin. The intersection number has been given in Section 3.2, where the symmetry differs only by the permutation of the variables x and y. Milnor numbers must satisfy $\Sigma\mu_i = \mu$. Since the Milnor number of the singularity Z_{11} is 11 and the contribution from the $\alpha = 0$ component is 6, then the $\alpha \neq 0$ component must contribute a total Milnor number of 5. Critical point on this component have $x = 0$ and y the root of a degree 5 polynomial. This means the critical points on this component occur with multiplicity 5, and this singularity is therefore $5A_1$.



Let h_1, h_2 denote the Picard-Lefschetz operators. These satisfy $(h_1h_2)^3 = (h_2h_1)^3$, giving the relation

$$|U|^2 = 25.$$

We know that $U \in \mathbb{Z}\langle\varepsilon_5\rangle$. Since $|U|^2$ is itself a square, we take $U = 5$, giving the following Dynkin diagram. Moreover, each of the 5 Morse cycles corresponding to $5A_1$ intersects the D_6 cycle at isolated points.

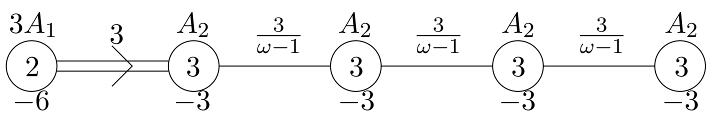


$Z_{11}|\mathbb{Z}_6$

The singularity $Z_{11}|\mathbb{Z}_6$ can be identified as

$$Z_{11}|\mathbb{Z}_6 \cong C_5^{(3)},$$

as seen in Section 3.1. Hence the diagram:



4.2.3 $E_{13} \ni x^3 + xy^5 + z^2$

The Coxeter element is $C(x,y,z) = (\omega x, \varepsilon_{15}^2 y, -z)$.

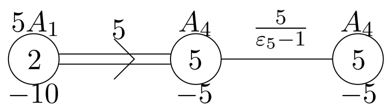
f	$g : x,y,z \mapsto$	$\lvert g\rvert$	versal monomials	notation
$x^3 + xy^5 + z^2 \in E_{13}$	C, C^2	$30, 15$	1	-
	C^3, C^6	$10, 5$	$1, x, y^5$	$E_{13}\lvert\mathbb{Z}_{10}$
	C^5, C^{10}	$6, 3$	$1, y^3, y^6, xy^2$	$E_{13}\lvert\mathbb{Z}_6$

$E_{13}|\mathbb{Z}_{10}$

The singularity $E_{13}|\mathbb{Z}_{10}$ may be identified as

$$E_{13}|\mathbb{Z}_{10} \cong C_3^{(5)},$$

by the boundary substitution $y_0 = y^5$. This gives the Dynkin diagram below. All information required to label the subdiagrams $B_2^{(5)}$ and $A_2^{(5)}$ is contained in Section 3.1.



$E_{13}|\mathbb{Z}_6$

For $E_{13}|\mathbb{Z}_6$ we have

$$\begin{aligned} F_g &= x^3 + xy^5 + z^2 + \delta xy^2 + \gamma y^6 + \beta y^3 + \alpha \\ F_{g,x} &= 3x^2 + y^5 + \delta y^2 \\ F_{g,y} &= 5xy^4 + 2\delta xy + 6\gamma y^5 + 3\beta y^2. \end{aligned}$$

If $y = 0$ at a critical point, we find also that $x = 0$ giving the discriminant component with equation $\Sigma_1 = \{\alpha = 0\}$. The normal form of a function germ corresponding to this component is

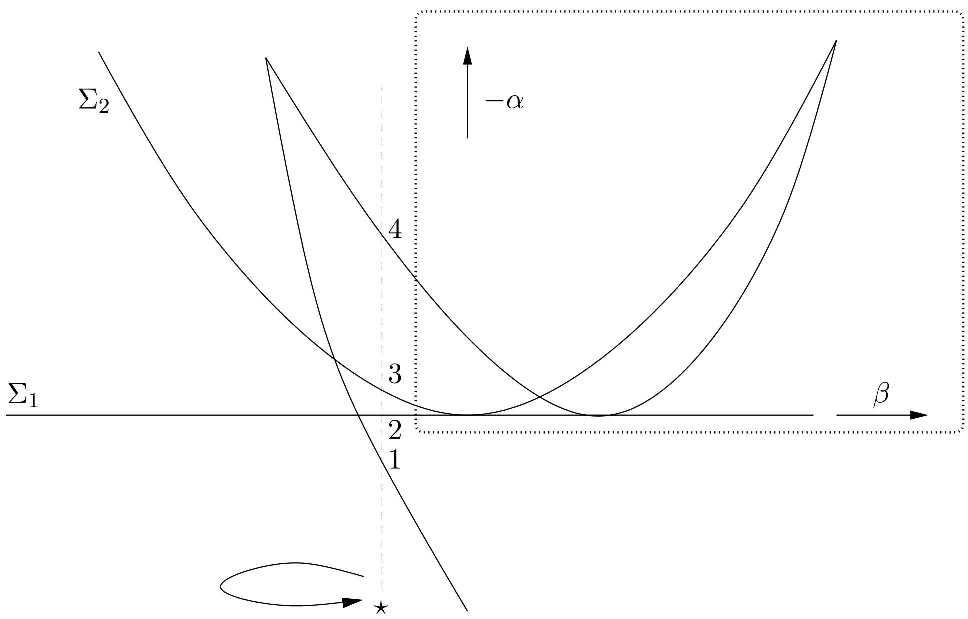
$$F_g|_{\alpha=0} \sim x^3 + y^3 + z^2,$$

a critical point of type D_4. If $y \neq 0$ at a critical point, we may eliminate x, y and z from the above system of equations to get the following equation for the other component of the discriminant.

$$\begin{aligned}\Sigma_2 \;=\; & \{-13500\beta\gamma^2\alpha^2 + 729\beta^4\delta\gamma + 729\beta^4\gamma^3 + 16\delta^6\gamma^3 - 216\delta^3\beta^3 \\ & -16d^6\beta + 16\delta^7\gamma + 3125\alpha^3 - 729\beta^5 + 216\delta^4\beta^2\gamma + 216\delta^3\beta^2\gamma^3 \\ & +4125\delta^2\gamma\alpha^2 - 5625\delta\beta\alpha^2 - 5832\beta^2\gamma^4\alpha + 6075\beta^3\gamma\alpha \\ & +2700\delta^2\beta^2\alpha + 864\delta^3\gamma^4\alpha + 888\delta^4\gamma^2\alpha + 16200\gamma^3\delta\alpha^2 - 5670\beta^2\gamma^2\delta\alpha \\ & -2592\delta^2\beta\gamma^3\alpha - 3420\delta^3\beta\gamma\alpha + 11664\gamma^5\alpha^2 + 16\delta^5\alpha = 0\}.\end{aligned}$$

Critical points on this component have multiplicity 3 by considering the symmetry g, and a generic line in the discriminant intersects this component at 3 points. So these critical point must be of type $3A_1$ to satisfy $\Sigma\mu_i = \mu$.

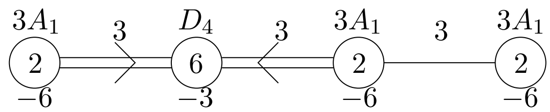
A generic section of the discriminant is given below, taken with sufficiently large $\gamma = \delta < 0$.



The majority of the Dynkin diagram for this singularity can be computed using the adjacency

$$E_{13}|\mathbb{Z}_6 \to J_{10}|\mathbb{Z}_3,$$

details of which can be found in Section 3.4.3. A generic 2 dimensional section of the discriminant of $J_{10}|\mathbb{Z}_3$ is boxed in the above figure.



Name the generators h_3, h_2, h_4, h_1 from left to right, where h_i denotes the Picard-Lefschetz operator induced by a loop in a generic line (dashed in the diagram) going around the intersection labelled i with the discriminant.The numbering is the order in which we loop around in the discriminant in the line from the base point $\star$ in the anticlockwise direction. The generators h_2, h_3, h_4 generate the group coming from $J_{10}|\mathbb{Z}_3$. The final generator h_1 commutes with h_2 and h_3, and from the generic section of the discriminant we see that it satisfies $h_1h_4h_1 = h_4h_1h_4$. The subdiagram for h_1, h_4 is that of $3A_2$, so the intersection number between the cycles must be 3 according to Section 3.2.

4.2.4 $Q_{10} \ni x^2z + y^3 + z^4$

The Coxeter element is $C(x, y, z) = (\varepsilon_8^3 x, \omega y, iz)$.

f	$g =$	$\lvert g\rvert$	versal monomials	notation
$x^2z + y^3 + z^4 \in Q_{10}$	C	24	1	-
	C^2	12	$1, z^2$	$Q_{10}\lvert\mathbb{Z}_{12}$
	C^3	8	$1, y$	$Q_{10}\lvert\mathbb{Z}_8$
	C^4	6	$1, z, z^2, z^3$	$Q_{10}\lvert\mathbb{Z}_6$
	C^6	4	$1, y, z^2, yz^2$	$Q_{10}\lvert\mathbb{Z}_4$
	C^8	3	$1, z, x, z^2, z^3$	$Q_{10}\lvert\mathbb{Z}_3$

$\underline{Q_{10}|\mathbb{Z}_{12}}$

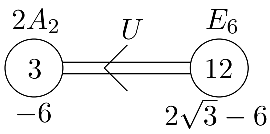
For $Q_{10}|\mathbb{Z}_{12}$ we have

$$\begin{aligned}
F_g &= x^2z + y^3 + z^4 + \beta z^2 + \alpha \\
F_{g,x} &= 2xz \\
F_{g,y} &= 3y^2 \\
F_{g,z} &= x^2 + 4z^3 + 2\beta z.
\end{aligned}$$

So $y = 0$ at a critical point. Assuming also that $z = 0$, we deduce $\alpha = 0$. The normal form of a function germ on this component is

$$F_g|_{\alpha=0} \sim x^4 + y^3 + z^2,$$

a critical point of type E_6, the self intersection number of such a cycle being described in Section 3.2. On the other hand, if $z \neq 0$ we find that $x = 0$, and singularities occur with multiplicity 2. To satisify $\Sigma\mu_i = \mu$ this singularity must be $2A_2$.



Let h_1, h_2 denote the Picard-Lefschetz operators. Since the generators satisfy $(h_1h_2)^2 = (h_2h_1)^2$, we calculate that $|U|^2 = 24$. We know also that $U \in \mathbb{Z}\langle\varepsilon_{12}\rangle$, and so must be of the form

$$U = k_1\varepsilon_{12} + k_2\overline{\varepsilon_{12}} + k_3\varepsilon_6 + k_4\overline{\varepsilon_6}.$$

The square of the modulus of this general element is given by

$$|U|^2 = k_1^2 + k_2^2 + k_3^2 + k_4^2 + k_1k_2 - k_3k_4 + (k_1k_3 + k_2k_4)\sqrt{3}.$$

Equating rational and irrational parts we find two conditions on our constants:

$$\begin{aligned} k_1^2 + k_2^2 + k_3^2 + k_4^2 + k_1k_2 - k_3k_4 &= 24 \\ k_1k_3 + k_2k_4 &= 0. \end{aligned}$$

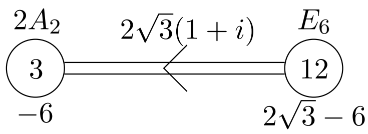
Putting these conditions into the computer program, we find there are twelve solutions for the integers k_i. One such solution is

$$k_1 = k_2 = k_3 = -k_4 = 2$$

corresponding to

$$U = 2\sqrt{3}(1+i).$$

Other solutions are of the form $\varepsilon_{12}^k U$ and correspond to the ambiguity in construction of the E_6 cycle. We complete our Dynkin diagram.

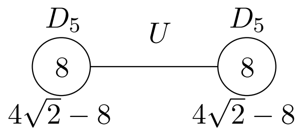


<u>$Q_{10}|\mathbb{Z}_8$</u>

For $Q_{10}|\mathbb{Z}_8$ we have

$$\begin{aligned} F_g &= x^2z + y^3 + z^4 + \beta y + \alpha \\ F_{g,x} &= 2xz \\ F_{g,y} &= 3y^2 + \beta \\ F_{g,z} &= x^2 + 4z^3 \end{aligned}$$

We see from $F_{g,x} = 0$ that either $x = 0$ or $z = 0$. Further, $F_{g,z}$ says that $x = 0$ if and only if $z = 0$, so we must have $x = z = 0$. This leaves an expression in y corresponding to a discriminant of type A_2. The discriminant component corresponds to a singularity of F_g of type D_5, which can be observed by considering monomials only involving x and z in the deformation. The Dynkin diagram starts to take shape.



Self intersection numbers shown have been stated in Section 3.2. Let h_1, h_2 denote the Picard-Lefschetz operators. Since we know that $h_1h_2h_1 = h_2h_1h_2$, we can calculate that $|U|^2 = 32 - 16\sqrt{2}$. Since $U \in \mathbb{Z}\langle\varepsilon_8\rangle$, it must be of the form

$$U = k_1 + k_2 i + k_3\varepsilon_8 + k_4\overline{\varepsilon_8}.$$

The square of the modulus of this general element is

$$|U|^2 = k_1^2 + k_2^2 + k_3^2 + k_4^2 + (k_1k_3 + k_1k_4 + k_2k_3 - k_2k_4)\sqrt{2}.$$

Equating rational and irrational parts we find two conditions on our constants:

$$\begin{aligned} k_1^2 + k_2^2 + k_3^2 + k_4^2 &= 32 \\ k_1k_3 + k_1k_4 + k_2k_3 - k_2k_4 &= -16. \end{aligned}$$

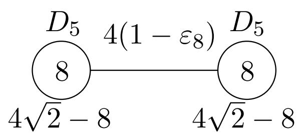
One of the 8 solutions to these equations is

$$k_1 = -k_3 = 4, k_2 = k_4 = 0,$$

corresponding to

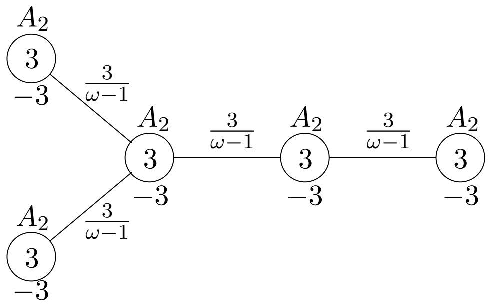
$$U = 4(1 - \varepsilon_8).$$

Other solutions are $\varepsilon_8^k U$, and we recall the ambiguity that a vanishing χ-cycle may be chosen up to multiplication by powers of χ. We complete our Dynkin diagram.



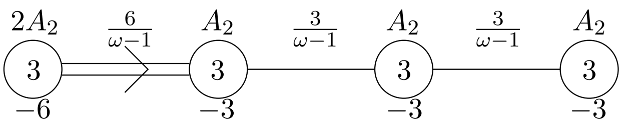
<u>$Q_{10}|\mathbb{Z}_3$</u>

The singularity $Q_{10}|\mathbb{Z}_3 \cong D_5^{(3)}$ by the boundary substitution $y_0 = y^3$ in the normal form $f \in Q_{10}$.



$\underline{Q_{10}|\mathbb{Z}_6}$

The singularity $Q_{10}|\mathbb{Z}_6$ is a folding of the above singularity $Q_{10}|\mathbb{Z}_3$, similar to getting the Dynkin diagram for C_4 from that of D_5, and can also be found by considering the adjacency $Q_{10}|\mathbb{Z}_6 \to C_3^{(3,3)}$, which appears as a symmetry of P_8. For details see Section 3.4.1.

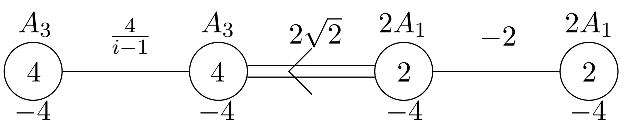


$\underline{Q_{10}|\mathbb{Z}_4}$

For $Q_{10}|\mathbb{Z}_4$ we have

$$
\begin{aligned}
F_g &= x^2z + y^3 + z^4 + \delta yz^2 + \gamma z^2 + \beta y + \alpha \\
F_{g,x} &= 2xz \\
F_{g,y} &= 3y^2 + \delta y^2 + \beta \\
F_{g,z} &= x^2 + 4z^3 + 2\delta yz + 2\gamma z.
\end{aligned}
$$

At any critical point $x = 0$. This leaves a deformation of $f|_{x=0} = y^3 + z^4$ which immediately gives us a discriminant of type F_4. We use the adjacency $Q_{10}|\mathbb{Z}_4 \to D_5|\mathbb{Z}_4 \cong B_2^{(4)}$, details of which are given in [12]. Intersection numbers can be read from Section 3.1. The diagram for $B_2^{(4)}$ is extended uniquely to the diagram for $Q_{10}|\mathbb{Z}_4$.



4.2.5 $E_{14} \ni x^3 + y^8 + z^2$

The Coxeter element is $C(x, y, z) = (\omega x, \varepsilon_8 y, -z)$.

f	$g =$	$\lvert g \rvert$	versal monomials	notation
$x^3 + y^8 + z^2 \in E_{14}$	C	24	1	-
	C^2	12	$1, y^4$	$E_{14}\lvert\mathbb{Z}_{12}$
	C^3	8	$1, x$	$E_{14}\lvert\mathbb{Z}_8$
	C^4	6	$1, y^2, y^4, y^6$	$E_{14}\lvert\mathbb{Z}_6$
	C^6	4	$1, x, y^4, xy^4$	$E_{14}\lvert\mathbb{Z}_4$
	C^8	3	$1, y, y^2, y^3, y^4, y^5, y^6$	$E_{14}\lvert\mathbb{Z}_3$

<u>$E_{14}|\mathbb{Z}_{12}$</u>

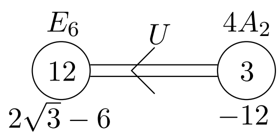
For $E_{14}|\mathbb{Z}_{12}$ we have

$$\begin{aligned} F_g &= x^3 + y^8 + z^2 + \beta y^4 + \alpha \\ F_{g,x} &= 3x^2 \\ F_{g,y} &= 8y^7 + 4\beta y^3. \end{aligned}$$

By considering the variable y we see the discriminant is of type B_2. For the component with equation $\alpha = 0$, the deformation has normal form

$$F_g|_{\alpha=0} \sim x^3 + y^4 + z^2,$$

a singularity of type E_6. On the remaining component, singularities occur with multiplicity 4. To satisfy $\Sigma\mu_i = \mu$, singularities corresponding to this component must be of type $4A_2$.



The self-intersection numbers are given in Section 3.2. Let h_1, h_2 denote the Picard-Lefschetz operators. The braiding relation on the generators

$(h_1h_2)^2 = (h_2h_1)^2$ gives the condition that $|U|^2 = 48$. Using the fact that $U \in \mathbb{Z}\langle\varepsilon_{12}\rangle$, U must have the form

$$U = k_1\varepsilon_{12} + k_2\overline{\varepsilon_{12}} + k_3\varepsilon_6 + k_4\overline{\varepsilon_6},$$

where the coefficients satisfy the relations

$$\begin{aligned} k_1^2 + k_2^2 + k_3^2 + k_4^2 + k_1k_2 - k_3k_4 &= 48 \\ k_1k_3 + k_2k_4 &= 0. \end{aligned}$$

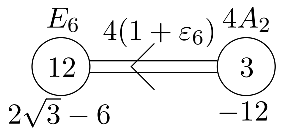
One solution is

$$k_1 = k_2 = 0, k_3 = 8, k_4 = 4,$$

corresponding to

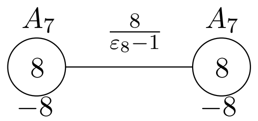
$$U = 4(1 + \varepsilon_6).$$

All other solutions are $\varepsilon_{12}^k U$, reflecting the ambiguity in the choice of the E_6 cycle. We complete our Dynkin diagram.



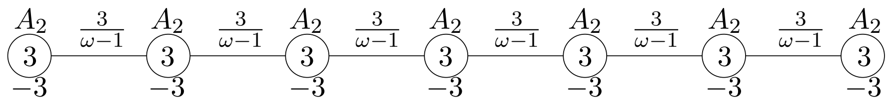
<u>$E_{14}|\mathbb{Z}_8$</u>

The singularity $E_{14}|\mathbb{Z}_8 \cong A_2^{(8)}$ by the boundary substitution $y_0 = y^8$ in the normal form $f \in E_{14}$. For details see Section 3.1.



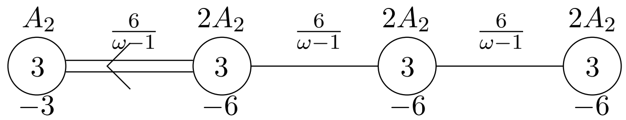
<u>$E_{14}|\mathbb{Z}_3$</u>

The singularity $E_{14}|\mathbb{Z}_3 \cong A_7^{(3)} \to E_{12}|\mathbb{Z}_3 \cong A_6^{(3)}$, so we construct the Dynkin diagram.



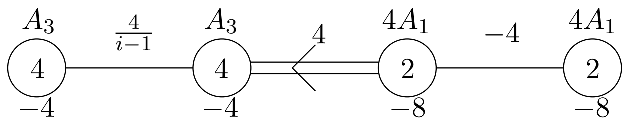
<u>$E_{14}|\mathbb{Z}_6$</u>

The singularity $E_{14}|\mathbb{Z}_6$ is a folding of $E_{14}|\mathbb{Z}_3$. We also note the adjacency $E_{14}|\mathbb{Z}_6 \to E_6|\mathbb{Z}_6 \cong B_2^{(3,3)}$, which can be seen as a subdiagram. Details of the latter can be found in Section 3.3.



<u>$E_{14}|\mathbb{Z}_4$</u>

The singularity $E_{14}|\mathbb{Z}_4 \cong F_4^{(4)}$ by the boundary substitution $y_0 = y^4$ in the normal form $f \in E_{14}$. We have an adjacency $E_{14}|\mathbb{Z}_4 \to A_7|\mathbb{Z}_4 \cong B_2^{(4)}$, details of which can be found in Section 3.3 giving the B_2 type subdiagram and we extend it in the unique way by adding simple edges.



4.2.6 $Z_{12} \ni x^3y + xy^4 + z^2$

The Coxeter element is $C = (\varepsilon_{11}x, \varepsilon_{11}^8 y, -z)$. All splitting symmetries of this singularity have one parameter g-versal deformations.

f	$g : x, y, z \mapsto$	$\lvert g\rvert$	versal monomials	notation
$x^3y + xy^4 + z^2 \in Z_{12}$	C	22	1	-

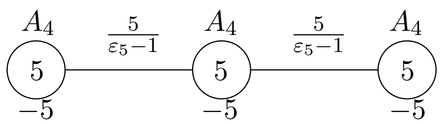
4.2.7 $W_{12} \ni x^4 + y^5 + z^2$

The Coxeter element is $C(x,y,z) = (ix, \varepsilon_5 y, -z)$. All symmetries given by powers of C are included in the table below. Since $\Delta = 40 = 2N$, we have other symmetries which are powers of C composed with $\iota_z(x,y,z) = (x,y,-z)$. Since the singularity is stably equivalent to a function of two variables and the involution ι_z affects only the third variable, these are omitted from the table.

f	$g =$	$\lvert g\rvert$	versal monomials	notation
$x^4 + y^5 + z^2 \in W_{12}$	C	20	1	-
	C^2	10	$1, x^2$	$W_{12}\lvert\mathbb{Z}_{10}$
	C^4	5	$1, x, x^2$	$W_{12}\lvert\mathbb{Z}_5$
	C^5	4	$1, y, y^2, y^3$	$W_{12}\lvert\mathbb{Z}_4$

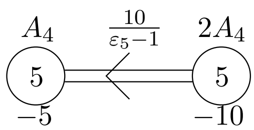
$W_{12}|\mathbb{Z}_5$

The singularity $W_{12}|\mathbb{Z}_5 \cong A_3^{(5)}$ by the boundary substitution $y_0 = y^5$ in the normal form $f \in W_{12}$.



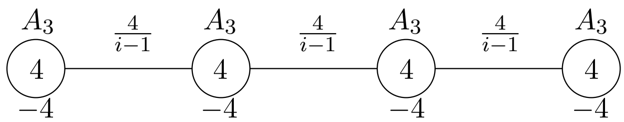
$W_{12}|\mathbb{Z}_{10}$

The singularity $W_{12}|\mathbb{Z}_{10}$ is a folding of $W_{12}|\mathbb{Z}_5$ by the involution $\iota_x(x,y,z) = (-x,y,z)$.



$W_{12}|\mathbb{Z}_4$

The singularity $W_{12}|\mathbb{Z}_4 \cong A_4^{(4)}$ by the boundary substitution $x_0 = x^4$ in the normal form $f \in W_{12}$.



4.2.8 $Q_{11} \ni x^2z + y^3 + yz^3$

The Coxeter element is $C(x, y, z) = (\varepsilon_{18}^7 x, \omega y, \varepsilon_9^2 z)$.

f	$g =$	$\|g\|$	versal monomials	notation
$x^2z + y^3 + yz^3 \in Q_{11}$	C	18	1	-
	C^2	9	1	-
	C^3, C^6	6	$1, y, z^3$	$Q_{11}\|\mathbb{Z}_6$

<u>$Q_{11}|\mathbb{Z}_6$</u>

We note the adjacency $Q_{11}|\mathbb{Z}_6 \to (P_8|\mathbb{Z}_6)'$, details of which are found in Section 3.4.1. Therefore the Dynkin diagram contains a subdiagram of the form $A_2^{(6)}$, in which vertices correspond to singularities of type D_4.

For $Q_{11}|\mathbb{Z}_6$ we have

$$\begin{aligned} F_g &= x^2z + y^3 + yz^3 + \gamma z^3 + \beta y + \alpha \\ F_{g,x} &= 2xz \\ F_{g,y} &= 3y^2 + z^3 + \beta \\ F_{g,z} &= x^2 + 3yz^2 + 3\gamma z^2 \end{aligned}$$

For $z \neq 0$, we find that $x = 0, y = -\gamma$ and z satisfies $z^3 + \beta + 3\gamma^2$ at zero level critical points. These correspond to singularities of type $3A_1$, since we already know two singularities with combined Milnor number 8, the total Milnor number must be 11, and z satisfies a cubic equation. The discriminantal component corresponding to this has equation

$$\Sigma_1 = \{\alpha - \beta\gamma - \gamma^3 = 0\}.$$

This is a smooth surface. We denote this component by Σ_1.

For $z = 0$, we find that $x = 0$, and γ is eliminated leaving us just to consider the variable y with parameters α, β satisfying an A_2 type discriminant relation, the cuspidal edge

$$\Sigma_1 = \{4\beta^3 + 27\alpha^2 = 0\}.$$

We consider the images Σ_1' and Σ_2' under the diffeomorphism

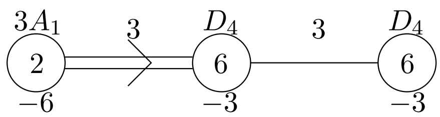
$$\begin{aligned} \alpha &\mapsto \alpha + \beta\gamma + \gamma^3 \\ \beta &\mapsto \beta \\ \gamma &\mapsto \gamma, \end{aligned}$$

giving

$$\Sigma_1' = \{\alpha = 0\},$$

$$\Sigma_2' = \{4\beta^3 + 27\alpha^2 + 54\alpha\beta\gamma + 54\alpha\gamma^3 + 27\gamma^2\beta^2 + 54\gamma^4\beta + 27\gamma^6 = 0\}$$

as the new equations of the discriminant. This is diffeomorphic to the C_3 type discriminant. The left two-vertex diagram comes from a symmetry of E_7 which was missed in [12].



4.2.9 $Z_{13} \ni x^3y + y^6 + z^2$

The Coxeter element is $C(x, y, z) = (\varepsilon_{18}^5 x, \varepsilon_6 y, -z)$. We have $\Delta = 36 = 2N$. All symmetries coming powers of C are included in the table below. All other symmetries are powers of C composed with $\iota_z(x, y, z) = (x, y, -z)$, and these are omitted.

f	$g =$	$\|g\|$	versal monomials	notation
$x^3y + y^6 + z^2 \in Z_{13}$	C	18	1	-
	C^2	9	$1, y^3$	$Z_{13}\|\mathbb{Z}_9$
	C^3	6	$1, y^2, y^4$	$Z_{13}\|\mathbb{Z}_6$
	C^6	3	$1, y, y^2, y^3, y^4, y^5$	$Z_{13}\|\mathbb{Z}_3$

$\underline{Z_{13}|\mathbb{Z}_9}$

For $Z_{13}|\mathbb{Z}_9$ we have

$$\begin{aligned} F_g &= x^3y + y^6 + z^2 + \beta y^3 + \alpha \\ F_{g,x} &= 3x^2y \\ F_{g,y} &= x^3 + 6y^5 + 3\beta y^2. \end{aligned}$$

The discriminant is easily seen to be of the B_2 type:

- its component $\alpha = 0$ corresponds to a singularity E_7 at the origin

$$F_g|_{\alpha=0} \sim x^3y + y^3 + z^2;$$

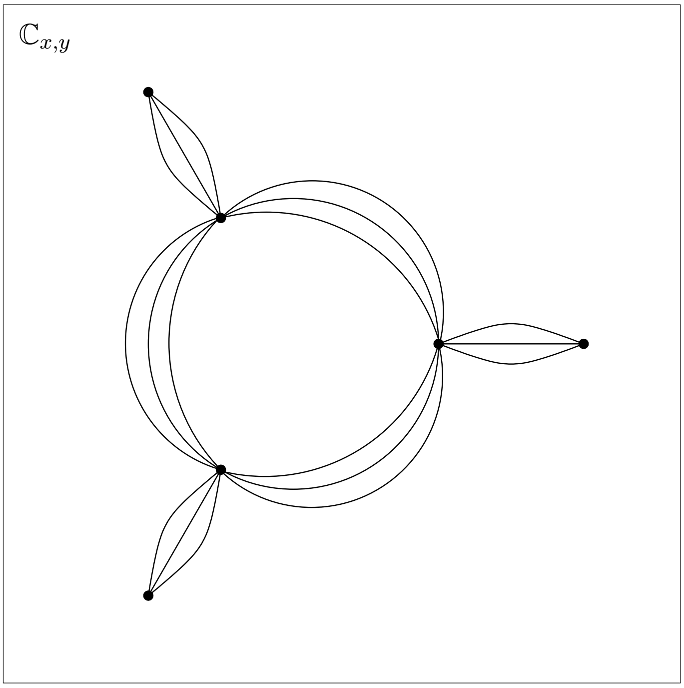
- and the component $\beta^2 - 4\alpha = 0$ is a $3A_2$ stratum.

In the latter case, each of the three A_2 singularities is $\mathbb{Z}_3$ symmetric with respect to $g^3(x,y,z) = (\overline{\omega}x, y, z)$ and has normal form $\tilde{x}^3 + \tilde{y}^2 + \tilde{z}^2$.

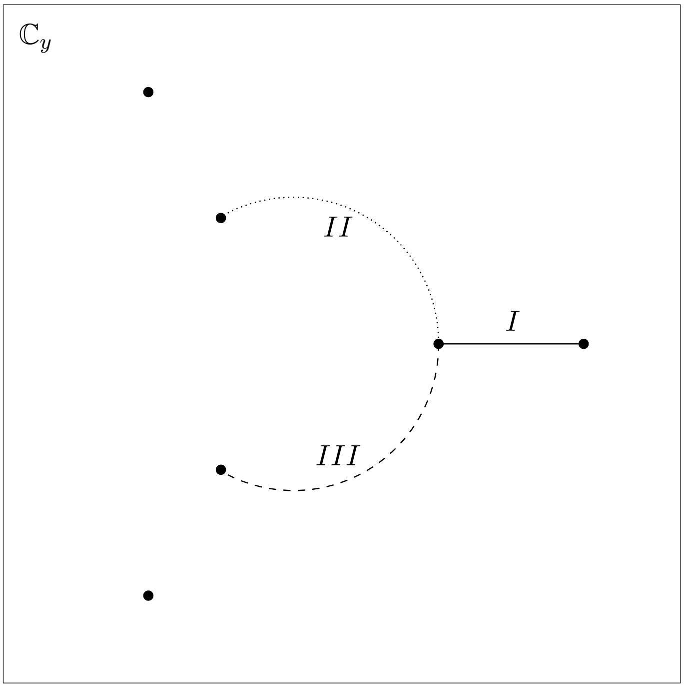
We take for a function corresponding to a generic marked point $\star$ in the base of g-versal deformation

$$F_\star = x^3y + (y^3 - 1)(y^3 - 8) + z^2.$$

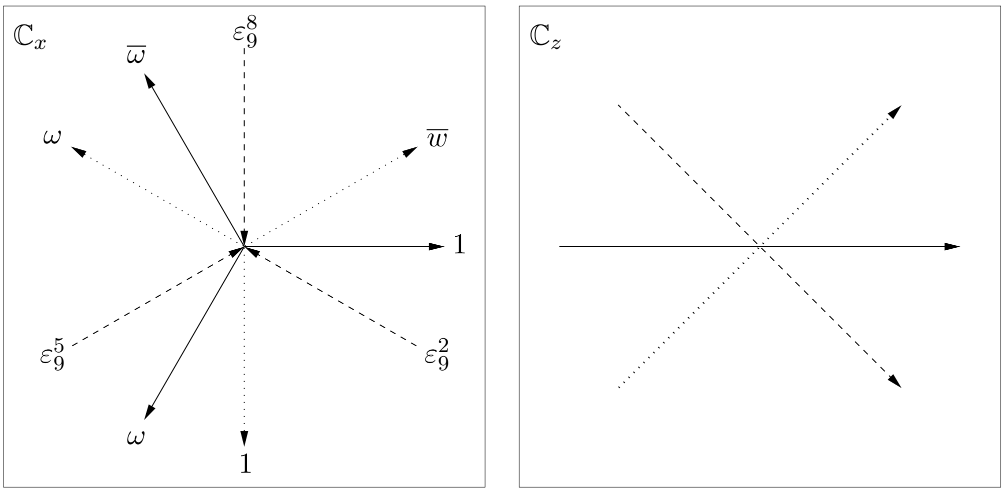
The curve $F_\star|_{z=0} = 0$ is a 3-cover of $\mathbb{C}_y$ with order 3 branching points at $y^3 = 1, y^3 = 8$, and with a puncture at the origin. It retracts onto the following configuration, with the nodes at the branching points.



In the 2 variable case, the vanishing E_7 χ-cycle is a linear combination of the 9 intervals along the circle and the $3A_2$ vanishing χ-cycle is a linear combination of the 9 others. The 3 variable case is a suspension of this. In particular, the self-intersection of the $3A_2$ χ-cycle is $3\times(-3) = -9$. Similarly, the self-intersection of each triple part of the 3 variable E_7 χ-cycle is -3. The rest of the intersection information about the two vanishing χ-cycles may be derived from the intersections at the branching point $y = 1$. So we consider the events over the arcs and interval in the picture below.



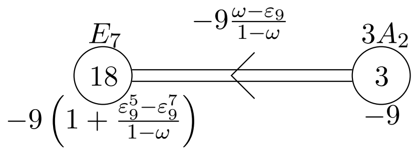
The orientations at $y = 1$ of the corresponding summands of the E_7 and $3A_2$ χ'-cycles are as follows.



We calculate the intersection numbers at $y = 1$.

$$\begin{aligned} \langle II, III\rangle &= 3\tfrac{\varepsilon_9^7}{1-\omega} & \langle I, II\rangle &= -3\tfrac{\omega}{1-\omega} \\ \langle III, II\rangle &= -3\tfrac{\varepsilon_9^5}{1-\omega} & \langle I, III\rangle &= 3\tfrac{\varepsilon_9}{1-\omega}. \end{aligned}$$

Taking appropriate sums of these numbers, we complete the Dynkin diagram.



<u>$Z_{13}|\mathbb{Z}_6$</u>

For $Z_{13}|\mathbb{Z}_6$ we have

$$\begin{aligned} F_g &= x^3y + y^6 + z^2 + \gamma y^4 + \beta y^2 + \alpha \\ F_{g,x} &= 3x^2y \\ F_{g,y} &= x^3 + 6y^5 + 4\gamma y^3 + 2\beta y \end{aligned}$$

.

If $\alpha = 0$ we have the normal form

$$F_g|_{\alpha=0} \sim x^6 + y^2 + z^2,$$

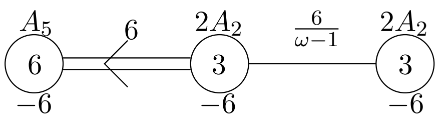
a singularity of type A_5.

When $\alpha \neq 0$ we find that $x = 0$. By considering the variable y we see that the discriminant is of type C_3, and singularities corresponding to the $\alpha \neq 0$ component of the discriminant occur with multiplicity 2. To satisfy $\Sigma\mu_i = \mu$ these must be of type $2A_2$.

We notice the adjacency

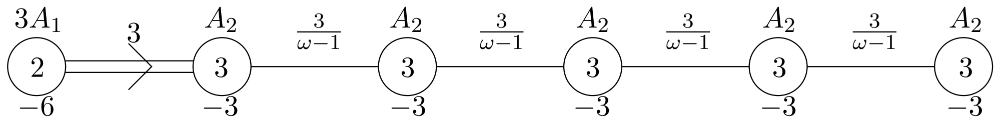
$$Z_{13}|\mathbb{Z}_6 \to X_9|\mathbb{Z}_6 \cong B_3^{(6,3)},$$

which appears as a subdiagram of the Dynkin diagram.



<u>$Z_{13}|\mathbb{Z}_3$</u>

The singularity $Z_{13}|\mathbb{Z}_3 \cong C_6^{(2,3)}$ by the boundary substitution $x_0 = x^3$ in the normal form $f \in Z_{13}$.



4.2.10 $S_{11} \ni x^2z + yz^2 + y^4$

The Coxeter element is $C(x, y, z) = (\varepsilon_{16}^5 x, iy, \varepsilon_8^3 z)$. Since $\Delta = 16 = N$, all symmetries are powers of C.

f	$g =$	$\lvert g\rvert$	versal monomials	notation
$x^2z + yz^2 + y^4 \in S_{11}$	C	16	1	-
	C^2	8	$1, y^2$	$S_{11}\lvert\mathbb{Z}_8$
	C^4	4	$1, y, y^2, z^2$	$S_{11}\lvert\mathbb{Z}_4$

<u>$S_{11}|\mathbb{Z}_8$</u>

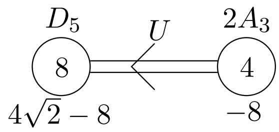
For $S_{11}|\mathbb{Z}_8$ we have

$$\begin{aligned} F_g &= x^2z + yz^2 + y^4 + \beta y^2 + \alpha \\ F_{g,x} &= 2xz \\ F_{g,y} &= z^2 + 4y^3 + 2\beta y \\ F_{g,z} &= x^2 + 2yz. \end{aligned}$$

The conditions $F_{g,x} = 0$ and $F_{g,z} = 0$ together imply that $x = 0$ at any zero-level critical point of the deformation. If $y = 0$, then $z = 0$ and $\alpha = 0$ gives the first component of the discriminant. This component corresponds to critical points of type D_5 (see Section 3.2). If $y \neq 0$, then from $F_{g,z}$ we must have $z = 0$. We are left with the conditions

$$\begin{aligned} y^4 + \beta y^2 + \alpha &= 0 \\ 4y^2 + 2\beta &= 0, \end{aligned}$$

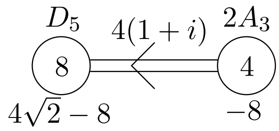
giving us the usual B_2 discriminant. Singularities of the deformation corresponding to the latter component of the discriminant are of type A_3 with multiplicity 2. We may start to construct the Dynkin diagram.



Using the relation $(h_1h_2)^2 = (h_2h_1)^2$ we get the condition that $|U|^2 = 32$. Similar calculations as in $Q_{10}|\mathbb{Z}_8$ show that we may take

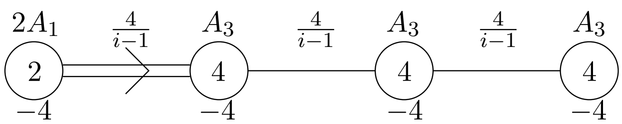
$$U = 4(1+i).$$

We may finish labelling our Dynkin diagram.



<u>$S_{11}|\mathbb{Z}_4$</u>

The ratio of degrees of the deformation parameters of $S_{11}|\mathbb{Z}_4$ coincides with that of C_4. Comparing the discriminants we can see that the unlabelled Dynkin diagrams also coincide. By noticing the adjacency $S_{11}|\mathbb{Z}_4 \to P_8|\mathbb{Z}_4 \cong C_3^{(4)}$ (see Section 3.4.1), we may construct our Dynkin diagram.



4.2.11 $W_{13} \ni x^4 + xy^4 + z^2$

The Coxeter element is $C(x,y,z) = (ix, \varepsilon_{16}^3 y, -z)$. All symmetries given by powers of C are included in the table below, and powers of C composed with $\iota_z(x,y,z) = (x,y,-z)$ are omitted for standard reasons.

f	$g =$	$\lvert g\rvert$	versal monomials	notation
$x^4 + xy^4 + z^2 \in W_{13}$	C	16	1	-
	C^2	8	$1, x^2$	$W_{13}\lvert\mathbb{Z}_8$
	C^4	4	$1, x, x^2, y^4$	$W_{13}\lvert\mathbb{Z}_4$

$W_{13}|\mathbb{Z}_8$

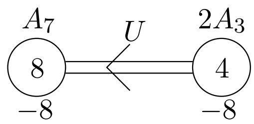
For $W_{13}|\mathbb{Z}_8$ we have

$$\begin{aligned} F_g &= x^4 + xy^4 + z^2 + \beta x^2 + \alpha \\ F_{g,x} &= 4x^3 + y^4 + 2\beta x \\ F_{g,y} &= 4xy^3. \end{aligned}$$

If $x = 0$, then $y = 0$ and we get a component of the discriminant with $\alpha = 0$. Singularities of the deformation corresponding to this component are type A_7. If $x \neq 0$ we still find that $y = 0$, and we get the conditions

$$\begin{aligned} x^4 + \beta x^2 + \alpha &= 0 \\ 4x^2 + 2\beta &= 0, \end{aligned}$$

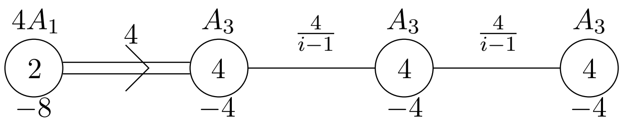
giving the usual B_2 type discriminant. Singularities of the deformation corresponding to the latter component of the discriminant are of type A_3 and of multiplicity 2. The self-intersection numbers are given in Section 3.2. The Dynkin diagram is given.



With the relation $(h_1h_2)^2 = (h_2h_1)^2$ on the generators h_1, h_2 and the fact that $U \in \mathbb{Z}\langle\varepsilon_8\rangle$, we use standard calculations to find $U = 4(1+i)(1+\varepsilon_8)$ and we complete our diagram.

$W_{13}|\mathbb{Z}_4$

The singularity $W_{13}|\mathbb{Z}_4 \cong C_4^{(4)}$ by the boundary substitution $y_0 = y^4$ in the normal form $f \in W_{13}$.



4.2.12 $Q_{12} \ni x^2z + y^3 + z^5$

The Coxeter element is $C(x, y, z) = (\varepsilon_5^2 x, \omega y, \varepsilon_5 z)$. If g is of the standard form $g(x, y, z) = (ax, by, cz)$, then the numbers a, b, c satisfy

$$a^2c = b^3 = c^5 = 1.$$

This system of equations has $2\mathrm{N} = 30$ solutions. Symmetries are compositions of powers of the classical monodromy with the involution

$$\iota_x(x, y, z) = (-x, y, z),$$

which corresponds to a symmetry of the Dynkin diagram of Q_{12} shown in Figure 4.2.

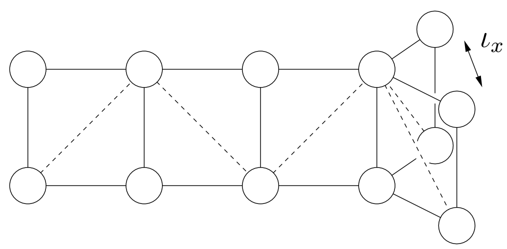


Figure 4.2: Dynkin diagram for Q_{12}

f	$g =$	$\lvert g\rvert$	versal monomials	notation
$x^2z + y^3 + z^5 \in Q_{12}$	$C, \iota_x C$	$15, 30$	1	-
	$C^3, \iota_x C^3$	$5, 10$	$1, y$	$Q_{12}\lvert\mathbb{Z}_{10}$
	$\iota_x C^5$	6	$1, z, z^2, z^3, z^4$	$Q_{12}\lvert\mathbb{Z}_6$
	C^5	3	$1, z, x, z^2, z^3, z^4$	$Q_{12}\lvert\mathbb{Z}_3$

$\underline{Q_{12}|\mathbb{Z}_{10}}$

For $Q_{12}|\mathbb{Z}_{10}$ we have

$$\begin{aligned} F_g &= x^2z + y^3 + z^5 + \beta y + \alpha \\ F_{g,x} &= 2xz \\ F_{g,y} &= 3y^2 + \beta \\ F_{g,z} &= x^2 + 5z^4. \end{aligned}$$

We find that at any zero-level critical point of F_g we have $x = z = 0$. We then have the system of equations

$$\begin{aligned} y^3 + \beta y + \alpha &= 0 \\ 3y^2 + \beta &= 0. \end{aligned}$$

Eliminating y gives the equation for the only component of the discriminant:

$$\left(\frac{\alpha}{2}\right)^2 = \left(\frac{-\beta}{3}\right)^3,$$

a standard A_2 type discriminant. The deformation F_g has singularities of type D_6 at discriminant points. The necessary self-intersection numbers have been described in Section 3.2. We begin to construct the Dynkin diagram.

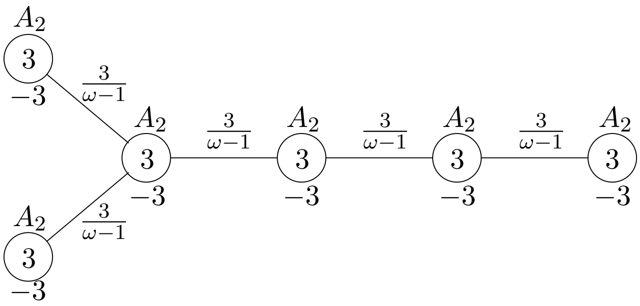
$$\begin{array}{ccc} D_6 & U & D_6 \\ 10 & \text{———} & 10 \\ \frac{5}{2}(\sqrt{5}-3) & & \frac{5}{2}(\sqrt{5}-3) \end{array}$$

Using the relation $h_1h_2h_1 = h_2h_1h_2$ we get the condition $|U|^2 = 25$. Since this is already a square number, we may take $U = 5$.

$$\begin{array}{ccc} D_6 & 5 & D_6 \\ 10 & \text{———} & 10 \\ \frac{5}{2}(\sqrt{5}-3) & & \frac{5}{2}(\sqrt{5}-3) \end{array}$$

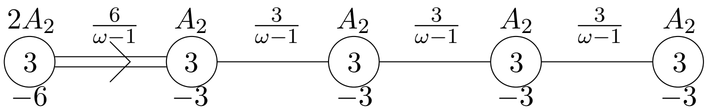
$\underline{Q_{12}|\mathbb{Z}_3}$

We can identify $Q_{12}|\mathbb{Z}_3 \sim D_6^{(3)}$ by the boundary substitution $y_0 = y^3$ in the normal form $f \in Q_{12}$. This is consistent with the adjacency $Q_{12}|\mathbb{Z}_3 \to Q_{10}|\mathbb{Z}_3$.



$\underline{Q_{12}|\mathbb{Z}_6}$

The singularity $Q_6|\mathbb{Z}_6$ is a folding of the above singularity $Q_{12}|\mathbb{Z}_6$. This is also consistent with the adjacency $Q_{12}|\mathbb{Z}_6 \to Q_{10}|\mathbb{Z}_6$. See Page 53.



4.2.13 $S_{12} \ni x^2z + yz^2 + xy^3$

The Coxeter element is $C(x, y, z) = (\varepsilon_{13}x, \varepsilon_{13}^4 y, \varepsilon_{13}^{11} z)$. Since $\Delta = 13 = N$ is prime, all splitting symmetries of this singularity have one parameter g-versal deformations.

f	$g =$	$\|g\|$	versal monomials	notation
$x^2z + yz^2 + xy^3 \in S_{12}$	C	13	1	-

4.2.14 $U_{12} \ni x^3 + y^3 + z^4$

The Coxeter element is $C(x, y, z) = (\omega x, \omega y, iz)$. The numbers a, b, c satisfy

$$a^3 = b^3 = c^4 = 1,$$

so we have

$$\Delta = \begin{vmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{vmatrix} = 36 = 3N.$$

Symmetries are compositions of powers of the Coxeter element with the map

$$\sigma(x, y, z) = (\omega x, \omega^2 y, z),$$

corresponding to a symmetry of the Dynkin diagram of U_{12} as shown in Figure 4.3.

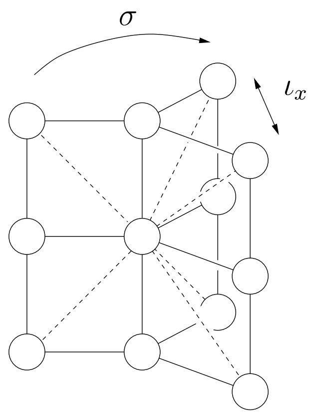


Figure 4.3: Dynkin diagram for U_{12}

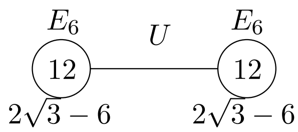
f	$g =$	$\|g\|$	versal monomials	notation
$x^3 + y^3 + z^4 \in U_{12}$	σC	12	$1, y$	$U_{12}\|\mathbb{Z}_{12}$
	C	12	1	-
	σC^2	6	$1, y, z^2, yz^2$	$U_{12}\|\mathbb{Z}_6$
	C^2	6	$1, z^2$	$(U_{12}\|\mathbb{Z}_6)'$
	σC^3	12	$1, xy$	$(U_{12}\|\mathbb{Z}_{12})'$
	C^3	4	$1, x, y, xy$	$U_{12}\|\mathbb{Z}_4$
	σC^4	3	$1, z, y, z^2, yz, yz^2$	$U_{12}\|\mathbb{Z}_3$
	C^4	3	$1, z, z^2$	$(U_{12}\|\mathbb{Z}_3)'$

$U_{12}|\mathbb{Z}_{12}$

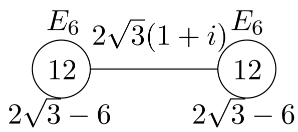
For $U_{12}|\mathbb{Z}_{12}$ we have

$$\begin{aligned} F_g &= x^3 + y^3 + z^4 + \beta y + \alpha \\ F_{g,x} &= 3x^2 \\ F_{g,y} &= 3y^2 + \beta \\ F_{g,z} &= 4z^3, \end{aligned}$$

giving $x = z = 0$ at any zero-level critical point of F_g. If we eliminate y from the remaining system of equations, we get a standard A_2 type discriminant, where points on the discriminant correspond to singularities of F_g of type E_6. Self-intersection numbers have been described in Section 3.2. We begin to construct the Dynkin diagram.

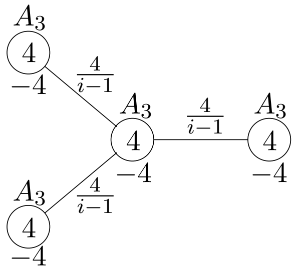


Using the relation $h_1h_2h_1 = h_2h_1h_2$, we find $|U|^2 = 24$. Standard calculations show that we may take $U = 2\sqrt{3}(1+i)$.



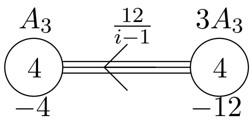
$U_{12}|\mathbb{Z}_4$

The singularity $U_{12}|\mathbb{Z}_4 \cong D_4^{(4)}$ by the boundary substitution $z_0 = z^4$ in the normal form $f \in U_{12}$.



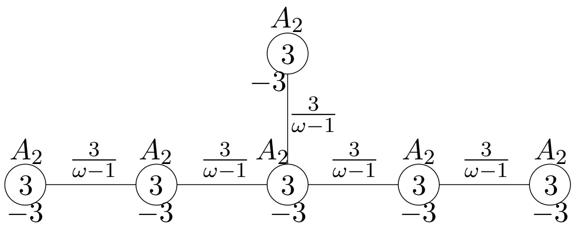
$(U_{12}|\mathbb{Z}_{12})'$

The singularity $(U_{12}|\mathbb{Z}_{12})'$ is an order 3 folding of $U_{12}|\mathbb{Z}_4$, analogous to the folding of D_4 to G_2.



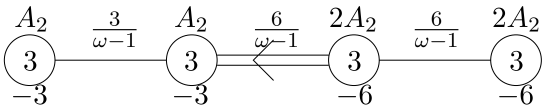
$U_{12}|\mathbb{Z}_3$

The singularity $U_{12}|\mathbb{Z}_3 \cong E_6^{(3)}$ by the boundary substitution $x_0 = x^3$ in the normal form $f \in U_{12}$.



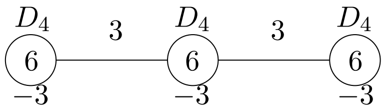
$U_{12}|\mathbb{Z}_6$

The singularity $U_{12}|\mathbb{Z}_6$ is a folding of $U_{12}|\mathbb{Z}_3$.



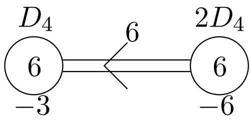
$(U_{12}|\mathbb{Z}_3)'$

For the singularity $(U_{12}|\mathbb{Z}_3)'$, we notice that the deformation monomials involve only the variable z, indicating that $x = y = 0$ at any component of the discriminant. The remaining system of equations involving z gives the standard A_3 swallowtail discriminant, with singularities of F_g of type D_4 at generic points of the discriminant. We find intersection numbers by using the adjacency $(U_{12}|\mathbb{Z}_3)' \to (P_8|\mathbb{Z}_6)'$. For details see Section 3.4.1.



$(U_{12}|\mathbb{Z}_6)'$

The singularity $(U_{12}|\mathbb{Z}_6)'$ is a simple folding of $(U_{12}|\mathbb{Z}_6)'$, analogous to the folding of A_3 to B_2.



4.2.15 $U_{12} \ni x^2y + y^3 + z^4$

The singularity U_{12} has two normal forms. Considering symmetries on the normal forms separately we get distinct results. All symmetries are described in the table, but only the monodromy group $(U_{12}|\mathbb{Z}_4)'$ has not appeared in the previous section.

As before, the Coxeter element is $C(x, y, z) = (\omega x, \omega y, iz)$. The numbers a, b, c must satisfy

$$a^2b = b^3 = c^4 = 1.$$

This system of equations has $2N = 24$ solutions. Symmetries are compositions of powers of the classical monodromy with the involution

$$\iota_x(x, y, z) = (-x, y, z),$$

which is a symmetry of the Dynkin diagram of U_{12} as shown in Figure 4.3.

f	$g =$	$\|g\|$	versal monomials	notation
$x^2y + y^3 + z^4 \in U_{12}$	$C, \iota_x C$	12	1	–
	$C^2, \iota_x C^2$	6	$1, z^2$	$(U_{12}\|\mathbb{Z}_6)'$
	$\iota_x C^3$	4	$1, y, y^2, xz^2$	$(U_{12}\|\mathbb{Z}_4)'$
	C^3	4	$1, x, y, y^2$	$(U_{12}\|\mathbb{Z}_4)''$
	$C^4, \iota_x C^4$	6	$1, z, z^2$	$(U_{12}\|\mathbb{Z}_3)'$

$\underline{(U_{12}|\mathbb{Z}_4)'}$

For $(U_{12}|\mathbb{Z}_4)'$ we have

$$\begin{aligned} F_g &= x^2y + y^3 + z^4 + \delta xz^2 + \gamma y^2 + \beta y + \alpha \\ F_{g,x} &= 2xy + \delta z^2 \\ F_{g,y} &= x^2 + 3y^2 + 2\gamma y + \beta \\ F_{g,z} &= 4z^2 + 2\delta xz. \end{aligned}$$

If $z = 0$ at a critical point, there are two cases to consider. The first case is when $y = 0$, in which case $x \neq 0$. This gives us a discriminantal component

$$\Sigma_1 = \{\alpha = 0\}.$$

The second case is when $z = 0$ but $y \neq 0$. This implies that $x = 0$, and the component of the discriminant appearing here has equation

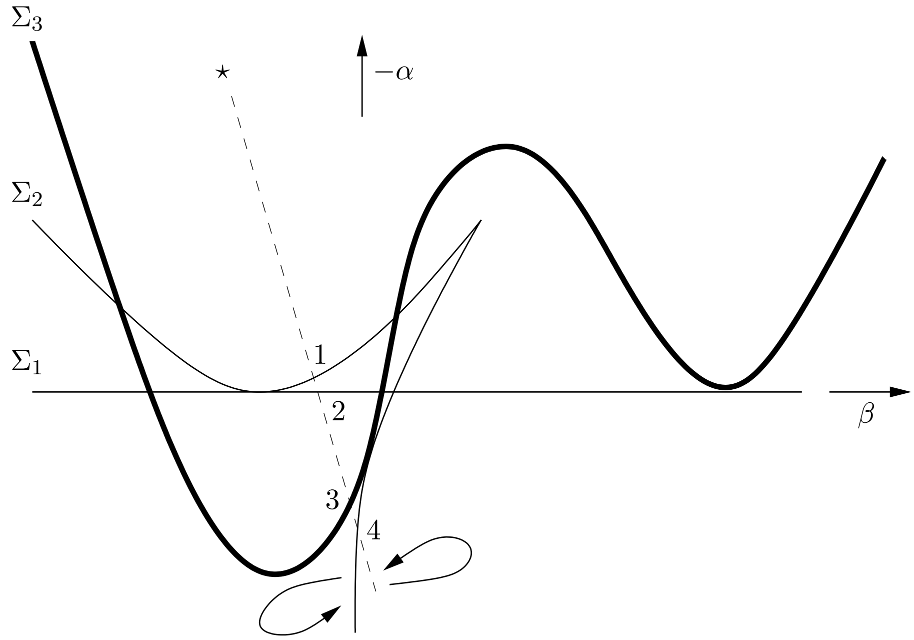
$$\Sigma_2 = \{27\alpha^2 - 18\alpha\beta\gamma - \beta^2\gamma^2 + 4\alpha\gamma^3 + 4\beta^3 = 0\}.$$

The union of these discriminantal components $\Sigma_1 \cup \Sigma_2$ is isomorphic to a B_3 type discriminant multiplied by the line in the direction of δ.

We consider finally the case $z \neq 0$. This implies that both $x \neq 0$ and $y \neq 0$, and gives a component with equation

$$\Sigma_3 = \{\delta^6 + 4\gamma\delta^4 + 16\beta\delta^2 + 64\alpha = 0\}.$$

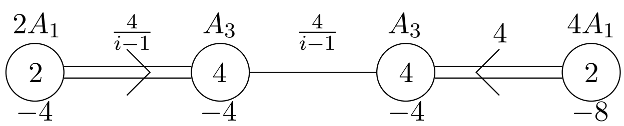
A generic two dimensional section can not be found by simply setting two variables to be constant. Rather, we take $\gamma < 0$ constant, and tilt slightly by setting $\delta = \gamma + \epsilon\beta$, for some small $\epsilon > 0$. In the figure below, Σ_3 is displayed in bold for distinction and the dashed line is the generic line from which we find our relations.



We notice the following adjacency:

$$(U_{12}|\mathbb{Z}_4)' \to J_{10}|\mathbb{Z}_4 \cong C_3^{(4)},$$

details of the latter being described in Section 3.4.3. This implies without calculation that critical points corresponding to Σ_2 are of type A_3. Since critical points corresponding to Σ_1 occur with multiplicity 2, these must be $2A_1$. Similarly, critical points corresponding to Σ_3 occur with multiplicity 4 and so must be $4A_1$. We denote the Picard-Lefschetz operators h_1, h_2, h_3, h_4 number them according to the anticlockwise order in which the corresponding simple loops leave $\star$. By the adjacency, the relations on h_1, h_3, h_4 are known. By moving the generic line around nearby non-generic points of the discriminant we can add the relations involving the generator h_2. The full set of relations is described in the Dynkin diagram below. Intersection numbers for the subdiagram corresponding to $J_{10}|\mathbb{Z}_4$ are described in Section 3.4.3, the remaining are described in Section 3.2. In this diagram the generators from right to left are h_2, h_1, h_4, h_3.



The weights of parameters are in the ratio $(1:2:4:6)$. This ratio is not seen as a ratio of weights of basic invariants for any group in the Shephard-Todd classification [21], and the skeleton of our group has not been seen as a linear complex reflection group.

$\underline{(U_{12}|\mathbb{Z}_4)''}$

The singularity $(U_{12}|\mathbb{Z}_4)''$ may be identified with $D_4^{(4)}$ by the boundary substitution $z_0 = z^4$, and has already been seen for $U_{12}|\mathbb{Z}_4$ on Page 71. This happens because U_{12}, like D_4, has two normal forms. In fact, we can think of U_{12} as being the direct sum of singularities

$$U_{12} = D_4 \oplus A_3$$

which has already been used to get the Dynkin diagram of U_{12}.

4.3 Classification of Equivariant Symmetries

The goal of the rest of this chapter is to list all smoothable equivariant symmetries of the unimodal singularities. According to Proposition 2.5, a necessary condition for an equivariant deformation to be smoothable is for it to have a linear term in its deformation. We list all deformations with a linear term and identify the non-smoothable deformations as they occur.

Proposition 4.1. *Let X be stably equivalent to a function germ in two variables. If the g-versal deformation is of the form*

$$F_g = x\psi(x, y, z, \lambda) + z^i,$$

where ψ is not constant, then F_g is not smoothable.

Proof. The curve $x\psi(x, y, \lambda) = 0$ in the (x, y)-plane is, for any λ, with singularities at the meeting points of its two component $x = 0$ and $\psi = 0$. Hence the corresponding surfaces in $\mathbb{C}^3$ are never smooth. $\square$

We regularly reference this proposition throughout the following classification to quickly identify non-smoothable deformations.

In the classification of invariant symmetries we used the formula $\Sigma\mu_i = \mu$ (see Page 34), where μ_i denotes the local Milnor number of the singularity related to the cycle corresponding to the Picard-Lefschetz operator h_i and μ is the Milnor number of the unimodal singularity we are considering. In short, the sum of local multiplicities is equal to the total Milnor number. Since equivariant deformations never include the constant monomial, this statement is generalised to

$$\Sigma\mu_i \frac{w_{f_i}}{w_{\alpha_i}} = \mu \frac{w_f}{w_\alpha},$$

where w_{f_i} is the quasi-degree of a function germ corresponding to the Picard-Lefschetz operator h_i, and w_{α_i} is the quasi-degree of the parameter α_i multiplying the linear term in the deformation (when this is defined uniquely). We also define w_f and w_α similarly for the function germ representing the unimodal singularity.

Throughout the rest of this chapter, following what has been said in Sections 2.1.2 and 2.1.3, we consider fractional powers of C. We do not consider any integer powers of C since these have already been described in the invariant classification.

We generalise the argument about the determinant of the matrix of exponents in the equivariant case. Consider the basic equivariant g_x of the meromorphic function f/x. The number of symmetries $(f/x) \circ g = f/x$ is equal to the absolute value of

$$\Delta_x = \begin{vmatrix} \alpha_1 - 1 & \beta_1 & \gamma_1 \\ \alpha_2 - 1 & \beta_2 & \gamma_2 \\ \alpha_3 - 1 & \beta_3 & \gamma_3 \end{vmatrix},$$

where $\alpha_i, \beta_i, \gamma_i$ are exponents of the system of equations in the corresponding invariant problem. If $\Delta_x = |g_x|$ then every invariant symmetry of f/x, that is every equivariant symmetry of f multiplying x by the same factor, is a power of g_x. We will also compare the basic equivariants to the Coxeter element.

4.3.1 $E_{12} \ni x^3 + y^7 + z^2$

Consider the meromorphic functions

$$\begin{aligned} f/x &= x^2 + y^7/x + z^2/x \\ f/y &= x^3/y + y^6 + z^2/y. \end{aligned}$$

The Coxeter element and basic equivariants are

$$\begin{aligned} C(x,y,z) &= (\omega x, \varepsilon_7 y, -z) \\ g_x(x,y,z;f) &= (-x, \varepsilon_{14}^3 y, -iz; -f) \\ g_y(x,y,z;f) &= (\varepsilon_{18}^7 x, \varepsilon_6 y, \varepsilon_{12}^7 z; \varepsilon_6 f), \end{aligned}$$

and we have $g_x = C^{\frac{3}{2}}$, $g_y = C^{\frac{7}{6}}$. The determinants of the matrices of exponents are

$$\Delta_x = \begin{vmatrix} 2 & 0 & 0 \\ -1 & 7 & 0 \\ -1 & 0 & 2 \end{vmatrix} = 28, \quad \Delta_y = \begin{vmatrix} 3 & -1 & 0 \\ 0 & 6 & 0 \\ 0 & -1 & 2 \end{vmatrix} = 36.$$

Since we have $\Delta_x = |g_x|$ and $\Delta_y = |g_y|$, all equivariant deformations preserving the monomial x or y are given by power of g_x or g_y respectively.

f	$g =$	$\lvert g\rvert$	versal monomials	notation
$x^3+y^7+z^2 \in E_{12}$	g_x	28	x	$-$
	g_y	18	y	$-$
	g_y^2, g_y^4	18,9	y, y^4	$E_{12}/\mathbb{Z}_9$
	g_y^3	12	y, y^3, y^5	$E_{12}/\mathbb{Z}_{12}$
	g_x^7, g_y^9	4	$y, x, y^3, xy^2, y^5, xy^4$	$E_{12}/\mathbb{Z}_4$

$\underline{E_{12}/\mathbb{Z}_9}$

For $E_{12}/\mathbb{Z}_9$ we have

$$\begin{aligned} F_g &= x^3 + y^7 + z^2 + \beta y^4 + \alpha y \\ F_{g,x} &= 3x^2 \\ F_{g_y} &= 7y^6 + 4\beta y^3 + \alpha. \end{aligned}$$

At any critical point we have $x = 0$. Considering the variable y we find the discriminant is of type B_2. For $\alpha = 0$ we have the equivalence

$$F_g|_{\alpha=0} \sim x^3 + y^4 + z^2,$$

a critical point of type E_6. For $\alpha \neq 0$ critical points occur with multiplicity 3. To satisfy $\Sigma \mu_i \frac{w_{f_i}}{w_{\alpha_i}} = \mu \frac{w_f}{w_\alpha}$, we make the following considerations.

We make a one-parameter deformation of a two variable representative function germ of E_{12} in the direction of y, that is

$$f = x^3 + y^7 + \alpha y.$$

For this quasi-homogeneous function we may take weights $w_f = 21, w_\alpha = 18$. The Milnor number of the singularity is $\mu = 12$ giving

$$\mu \frac{w_f}{w_\alpha} = 14.$$

The first discriminant component $\{\alpha = 0\}$ corresponds to singularities of type E_6, for which we also write a one-parameter deformation in the direction of y,

$$f_1 = x^3 + y^4 + \alpha_1 y.$$

In this case we may take $w_{f_1} = 12$, $w_{\alpha_1} = 9$ to make f_1 quasi-homogeneous. Then we have

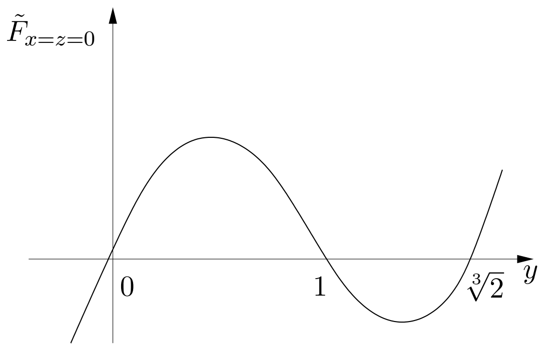
$$\mu_1 \frac{w_{f_1}}{w_{\alpha_1}} = 8.$$

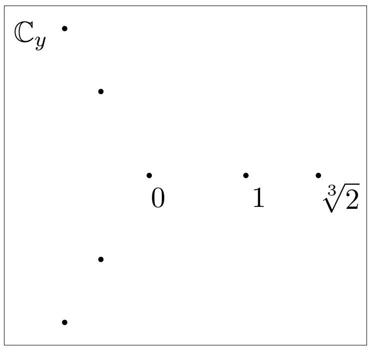
This leaves a contribution of 6 which must come from the other components of the discriminant. Since these occur with multiplicity 3, they must be of type $3A_2$.

In this y-axis $x = z = 0$, consider the deformation

$$\tilde{F}_{x=z=0} = y^7 - 3y^4 + 2y,$$

with its graph and the roots in the complex line $\mathbb{C}_y$ shown below.

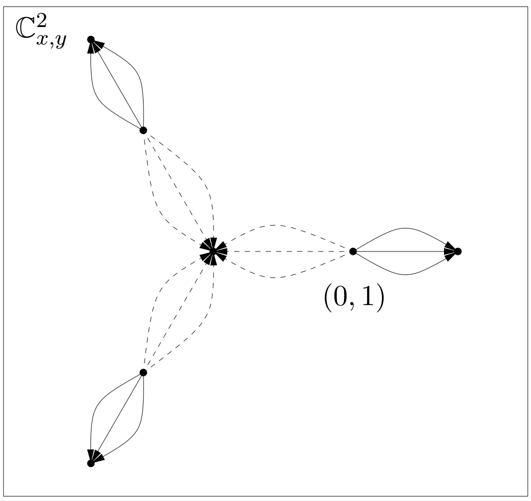




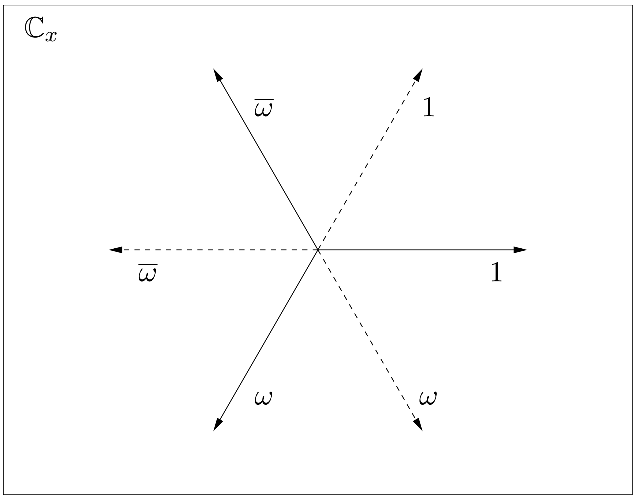
Adding the variable x^3 to give

$$\tilde{F}_{z=0} = x^3 + y^7 - 3y^4 + 2y$$

produces an order 3 ramification of $\mathbb{C}_y$ whose branching points are roots of $\tilde{F}_{x=z=0}$. A schematic picture of the cycles this produces are shown.

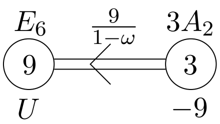


At the point $(x, y) = (0, 1)$ of this surface we choose x for the coordinate. So the meeting of the two triples is:

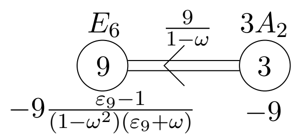


Locally, this is the same as the intersection of two $A_1^{(3)}$ $\chi = \omega$-cycles. Locally, this figure gives $\frac{3}{1-\omega}$. Addition of z^2, for this local intersection number, is similar to the stabilisation of the $\mathbb{Z}_3$-symmetric function $y^3 - 3y^2 + 2y + x_0(+z^2)$, $x_0 = x^3$. Since this local configuration is present at three branching points, we multiply this number by 3 to obtain the intersection number between cycles $e_1, e_2 : \langle e_1, e_2 \rangle = \frac{9}{1-\omega}$.

We start to construct the Dynkin diagram.

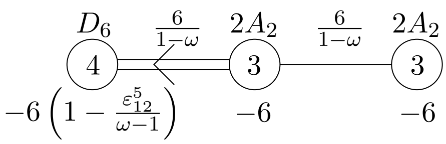


We have still to find the self-intersection number of the cycle corresponding to E_6. Instead of doing this directly, we use the fact that the generators satisfy the relation $(h_1 h_2)^2 = (h_2 h_1)^2$. The unknown U is the solution of a linear equation and thus the solution is unique.



$\underline{E_{12}/\mathbb{Z}_{12}}$

The singularity $E_{12}/\mathbb{Z}_{12}$ is adjacent to $B_2^{(4,3)}$, details of which are given in Section 3.3. Moreover, its deformation has discriminant of type B_3, so it remains only to extend the diagram.

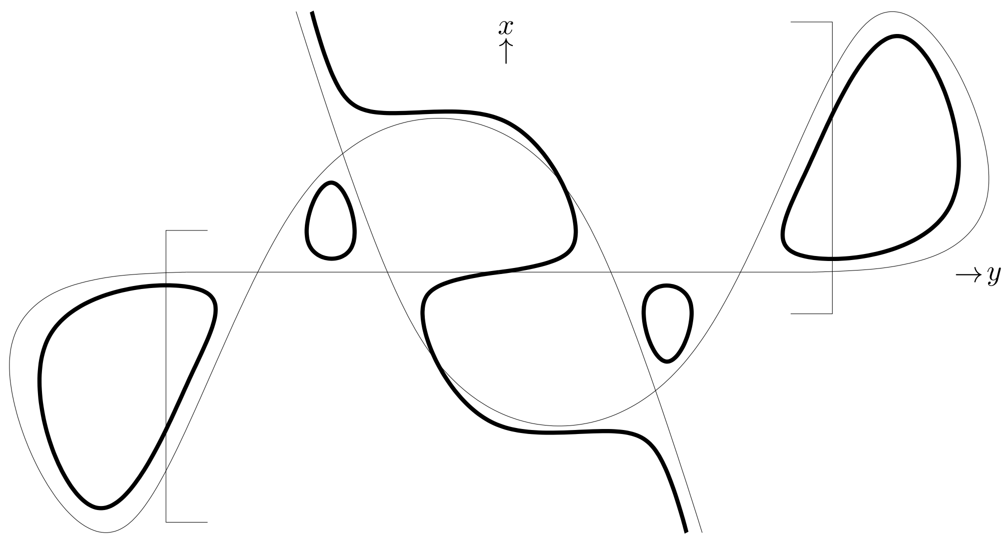


$\underline{E_{12}/\mathbb{Z}_4}$

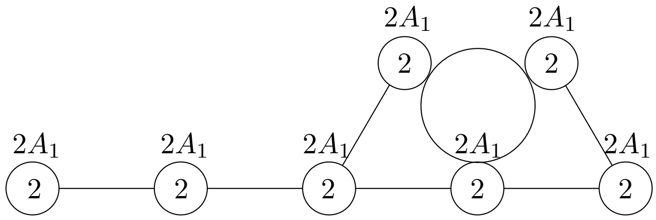
The singularity $E_{12}/\mathbb{Z}_4$ is adjacent $J_{10}/\mathbb{Z}_4$, details of which can be found in Section 3.4.3. A sabirification (see [12]) of a function germ corresponding to $J_{10}/\mathbb{Z}_4$ is given by

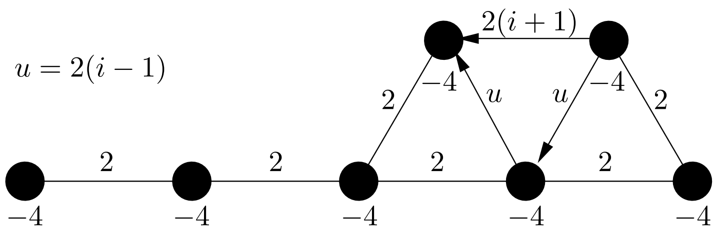
$$F_\alpha^J = x\left(x+y^2+y-1\right)\left(x-y^2+y+1\right)+\alpha y$$

where the zero-level set is given by F_0^J. The following figure is arranged as follows. Between the brackets [ ] the thin curve represents the graph $F_0^J = 0$, the thick $F_\alpha^J = 0$, for some small, real $\alpha \neq 0$.



Adding the monomial y^7 we define $F_\alpha = y^7 + F_\alpha^J$, which is a deformation of $E_{12}/\mathbb{Z}_4$, which modifies the above graph of $J_{12}/\mathbb{Z}_4$ by closing up the loops on the outer sides of the brackets [ ]. Using this, we claim there exists a sabirification of $E_{12}/\mathbb{Z}_4$ whose graph is diffeomorphic to that given in the figure, ignoring the brackets. Then the Dynkin diagram for $E_{12}/\mathbb{Z}_4$ is a simple extension of the diagram for $J_{10}/\mathbb{Z}_4$.





We denote the lower of the three generators connected by curved edges by h_4, and travelling clockwise the other two are h_7, h_6 respectively to match Table 5.1 on page 127. The corresponding χ'-cycles satisfy $e_4 = e_6 + e_7$.

4.3.2 $Z_{11} \ni x^3y + y^5 + z^2$

The Coxeter element and basic equivariants are given by

$$\begin{aligned} C(x,y,z) &= (\varepsilon_{15}^4 x, \varepsilon_5 y, -z) \\ g_x(x,y,z;f) &= (\varepsilon_{11}^4 x, \varepsilon_{11}^3 y, \varepsilon_{22}^{15} z; \varepsilon_{11}^4) \\ g_y(x,y,z;f) &= (\omega x, iy, \varepsilon_8^5 z; if). \end{aligned}$$

We have $g_x = C^{\frac{15}{11}}$, $g_y = C^{\frac{5}{4}}$. Since $\Delta_x = |g_x| = 22$ and $\Delta_y = |g_y| = 24$, all symmetries are powers of the basic equivariants.

f	$g =$	$\|g\|$	versal monomials	notation
$x^3y + y^5 + z^2 \in Z_{11}$	g_y, g_y^3	24,8	y	$-$
	g_y^2	12	y, y^3	$Z_{11}/\mathbb{Z}_{12}$
	g_x, g_x^2	22,11	x	$-$
	g_y^6	4	y, y^3, xy, xy^3	$Z_{11}/\mathbb{Z}_4$

The singularities $Z_{11}/\mathbb{Z}_{12}$ and $Z_{11}/\mathbb{Z}_4$ are not smoothable since in both cases the deformation is of the form

$$F_g = y\psi(x, y, \lambda) + z^2.$$

4.3.3 $E_{13} \ni x^3 + xy^5 + z^2$

The Coxeter element and basic equivariants are

$$\begin{aligned} C(x, y, z) &= \omega x, \varepsilon_{15}^2 y, -z) \\ g_x(x, y, z; f) &= (-x, \varepsilon_5 y, -iz; -f) \\ g_y(x, y, z; f) &= (\varepsilon_{13}^5 x, \varepsilon_{13}^2 y, \varepsilon_{26}^{15} z; \varepsilon_{13}^2 f). \end{aligned}$$

We have $g_x = C^{\frac{3}{2}}$, $g_y = C^{\frac{15}{13}}$. Since $\Delta_x = |g_x| = 20$ and $\Delta_y = |g_y| = 26$, all symmetries are powers of the basic equivariants.

f	$g =$	$\|g\|$	versal monomials	notation
$x^3 + xy^5 + z^2 \in E_{13}$	g_x	20	x	$-$
	g_y, g_y^2	26, 13	y	$-$
	g_x^5	4	x, xy, xy^2, xy^3	$E_{13}/\mathbb{Z}_4$

The singularity $E_{13}/\mathbb{Z}_4$ is not smoothable since its deformation is of the form

$$F_g = x\psi(x, y, \lambda) + z^2.$$

4.3.4 $Q_{10} \ni x^2z + y^3 + z^4$

The classical monodromy and basic equivariants are

$$\begin{aligned} C(x,y,z) &= (\varepsilon_8^3 x, \omega y, iz) \\ g_x(x,y,z;f) &= (\varepsilon_5^3 x, \varepsilon_{15}^8 y, \varepsilon_5^2 z; \varepsilon_5^3 f) \\ g_y(x,y,z;f) &= (\varepsilon_{16}^9 x, -y, \varepsilon_8^3 z; -f) \\ g_z(x,y,z;f) &= (-x, \varepsilon_9^4 y, \omega z; \omega f). \end{aligned}$$

We have $g_x = h^{\frac{8}{5}}$, $g_y = h^{\frac{3}{2}}$, $g_z = h^{\frac{4}{3}}$. Since the determinants of the matrices of exponents are equal to the order of the basic equivariants, all symmetries are powers of the basic equivariants.

f	$g =$	$\|g\|$	versal monomials	notation
$x^2z + y^3 + z^4 \in Q_{10}$	g_z, g_z^2	$18, 9$	z	–
	g_y	16	y	–
	g_x	15	x	–
	g_x^3	5	x, yz	$Q_{10}/\mathbb{Z}_5$

<u>$Q_{10}/\mathbb{Z}_5$</u>

The singularity $Q_{10}/\mathbb{Z}_5$ has deformation and partial derivatives:

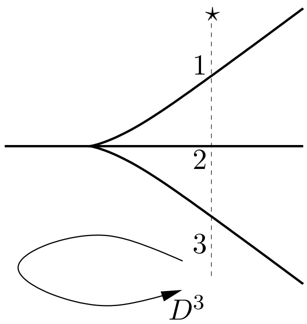
$$\begin{aligned} F_g &= x^2z + y^3 + z^4 + \beta yz + \alpha x \\ F_{g,x} &= 2xz + \alpha \\ F_{g,y} &= 3y^2 + \beta z \\ F_{g,z} &= x^2 + 4z^3 + \beta y. \end{aligned}$$

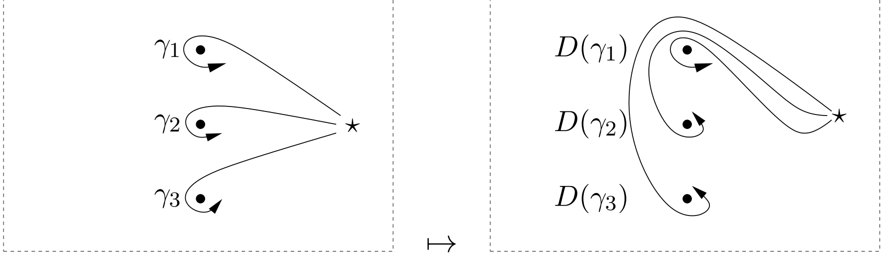
For $\alpha = 0$ we have

$$F_g|_{\alpha=0} \sim x^2z + y^3 + yz.$$

The Milnor number of this is 5 (by finding a basis for the local algebra, for example), and the presence of the non-degenerate quadratic form yz means this is a codimension 2 singularity. For these reasons, the critical point is of type A_5. For $\alpha \neq 0$ the discriminantal component is of type A_2. A generic line intersects this component twice, and singularities on this component occur with multiplicity 2. To satisfy $\Sigma\mu_i \frac{w_{f_i}}{w_{\alpha_i}} = \mu \frac{w_f}{w_\alpha}$, this component must correspond to singularities of type $5A_1$.

The discriminant is the union of a standard A_2 type cusp and a line tangent to the cusp. We will calculate the braiding relations for the generators of the fundamental group of the complement to the discriminant. Take a generic line in the complement to the discriminant transverse to the line $\alpha = 0$. In this line we may use β as a coordinate and identify the line with the space $\mathbb{C}_\beta$. Projecting the generic point $\star$ along this line gives a point in the space $\mathbb{C}_\alpha$. A operator D that moves a point continuously by $2\pi/3$ around the origin in $\mathbb{C}_\alpha$ induces a homotopy between a copy of the line over 1 and a copy of the line over ω. The operator D is defined in such a way that D^3 defines a closed path in $\mathbb{C}_\alpha$, the induced homotopy of which in the whole space $\mathbb{C}^2$ is just the identity. We watch what happens to the generators during this process to find relations.





Let $\bar{\gamma}_i$ denote the loop γ_i traversed in the opposite direction. The operator D may be written in terms of the generating loops as follows.

$$\begin{array}{rcl} D: & & \\ \gamma_1 & \mapsto & \bar{\gamma}_1\bar{\gamma}_2\gamma_3\gamma_2\gamma_1 \\ \gamma_2 & \mapsto & \bar{\gamma}_1\gamma_2\gamma_1 \\ \gamma_3 & \mapsto & \gamma_1 \end{array}$$

Note that this also implies that $D : \gamma_3\gamma_2\gamma_1 \mapsto \gamma_3\gamma_2\gamma_1$. We continue.

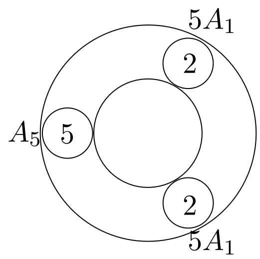
$$\begin{array}{rcl} D^2: & & \\ \gamma_1 & \mapsto & \bar{\gamma}_1\bar{\gamma}_2\bar{\gamma}_3\gamma_1\gamma_3\gamma_2\gamma_1 \\ \gamma_2 & \mapsto & \bar{\gamma}_1\bar{\gamma}_2\bar{\gamma}_3\gamma_2\gamma_3\gamma_2\gamma_1 \\ \gamma_3 & \mapsto & \bar{\gamma}_1\bar{\gamma}_2\gamma_3\gamma_2\gamma_1 \end{array}$$

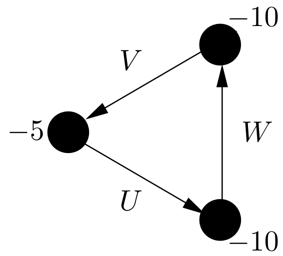
$$\begin{array}{rcl} D^3: & & \\ \gamma_1 & \mapsto & \bar{\gamma}_1\bar{\gamma}_2\bar{\gamma}_3\bar{\gamma}_1\bar{\gamma}_2\gamma_3\gamma_2\gamma_1\gamma_3\gamma_2\gamma_1 \\ \gamma_2 & \mapsto & \bar{\gamma}_1\bar{\gamma}_2\bar{\gamma}_3\bar{\gamma}_1\gamma_2\gamma_1\gamma_3\gamma_2\gamma_1 \\ \gamma_3 & \mapsto & \bar{\gamma}_1\bar{\gamma}_2\bar{\gamma}_3\gamma_1\gamma_3\gamma_2\gamma_1 \end{array}$$

Since the action of D^3 on the generating loops is homotopically equivalent to the identity, we deduce the braiding relations

$$\begin{aligned} \gamma_1\gamma_2\gamma_3\gamma_2\gamma_1 &= \gamma_3\gamma_2\gamma_1\gamma_3\gamma_2 \\ \gamma_1\gamma_3\gamma_2\gamma_1\gamma_2 &= \gamma_2\gamma_1\gamma_3\gamma_2\gamma_1 \\ \gamma_3\gamma_2\gamma_1\gamma_3 &= \gamma_1\gamma_3\gamma_2\gamma_1, \end{aligned}$$

where the first relation may be ignored since it is implied by the other two. We find that this group is isomorphic to the braid group obtained from taking the unique lift of the Shephard-Todd group G_{13} in the style of [7], and use the convention from that paper to draw the Dynkin diagram, where the generator in the bottom right is h_1, and travelling clockwise generators are h_1, h_2, h_3. We use unknowns U, V, W to denote the unknown intersection numbers.





Let e_1, e_2, e_3 denote the χ'-cycles corresponding to generators h_1, h_2, h_3. We may normalise e_2 according to the ambiguity in labelling chains up to power of $-\varepsilon_5$ so that the following relation holds for some constant K

$$e_1 + e_3 - Ke_2 = 0.$$

Taking the intersection of this condition with each cycle, we find the system of equations

$$\langle e_1, e_i \rangle + \langle e_3, e_i \rangle - K \langle e_2, e_i \rangle = 0$$

for $i = 1, 2, 3$. Substituting in the unknowns U, V, W, the system becomes

$$\begin{aligned} -10 + w - KU &= 0 \\ u + V + 5K &= 0 \\ W - 10 - Kv &= 0, \end{aligned}$$

where we use the notation $\overline{U} = u$, and so on.

The unknowns U, W (and therefore u, w) can be eliminated from this system of equations by

$$\begin{aligned} U &= -v - 5\overline{K} \\ W &= 10 + Kv, \end{aligned}$$

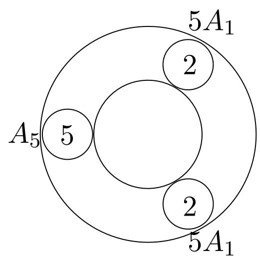
leaving the condition on the remaining unknowns

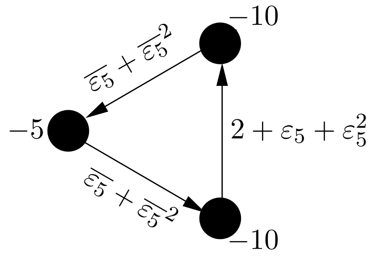
$$9 - \overline{K}V - Kv + 10|K|^2 = 0. \tag{4.1}$$

In the chart e_1, e_2, we write our generators explicitly and use the relations

$$\begin{aligned} h_1h_3h_2h_1h_2 - h_2h_1h_3h_2h_1 &= 0 \\ h_3h_2h_1h_3 - h_1h_3h_2h_1 &= 0. \end{aligned}$$

We see by taking resultants of entries in these matrices to eliminate V that each matrix relation yields one scalar equation. Let these be S and T. Taking resultant(S, T, v) gives a degree 4 equation on $|K|^2$. This has four solutions: 1, a negative solution, and two complicated solutions. Working backwards from the complicated solution shows that they violate the hyperbolicity condition (the determinant of the intersection matrix must be negative), and we must therefore choose $K = 1$. This provides all of the other intersection numbers.





As we calculated, the cycles satisfy the relation $e_1 + e_3 = e_2$.

4.3.5 $E_{14} \ni x^3 + y^8 + z^2$

The Coxeter element and basic equivariants are given by

$$\begin{aligned} C(x,y,z) &= (\omega x, \varepsilon_8 y, -z) \\ g_x(x,y,z;f) &= (-x, \varepsilon_{16}^3 y, -iz; -f) \\ g_y(x,y,z;f) &= (\varepsilon_{21}^8 x, \varepsilon_7 y, \varepsilon_{14} z; \varepsilon_7 f). \end{aligned}$$

We have $g_x = C^{\frac{3}{2}}$, $g_y = \iota_z C^{\frac{8}{7}}$. Symmetries coming from powers of the basic equivariants are given in the table below. The determinant of exponents $\Delta_x = 32 = 2|g_x|$, and extra symmetries are composition of powers of g_x with the involution ι_z. Since this singularity is stably equivalent to a function germ of two variables, z being the third, we ignore such symmetries. We also observe $\Delta_y = 42 = |g_y|$, and consider symmetries coming from powers of g_y.

f	$g =$	$\lvert g\rvert$	versal monomials	notation
$x^3 + y^8 + z^2 \in E_{14}$	g_y, g_y^2	$42, 21$	y	$E_{14}/\mathbb{Z}_{21}$
	g_x	16	x	$E_{14}/\mathbb{Z}_{16}$
	g_y^3, g_y^6	$14, 7$	y, xy^3	$E_{14}/\mathbb{Z}_7$

$E_{14}/\mathbb{Z}_7$

The singularity $E_{14}/\mathbb{Z}_7$ has deformation and partial derivatives

$$\begin{aligned} F_g &= x^3 + y^8 + z^2 + \beta x y^3 + \alpha y \\ F_{g,x} &= 3x^2 + \beta y^3 \\ F_{g,y} &= 8y^7 + 3\beta x y^2. \end{aligned}$$

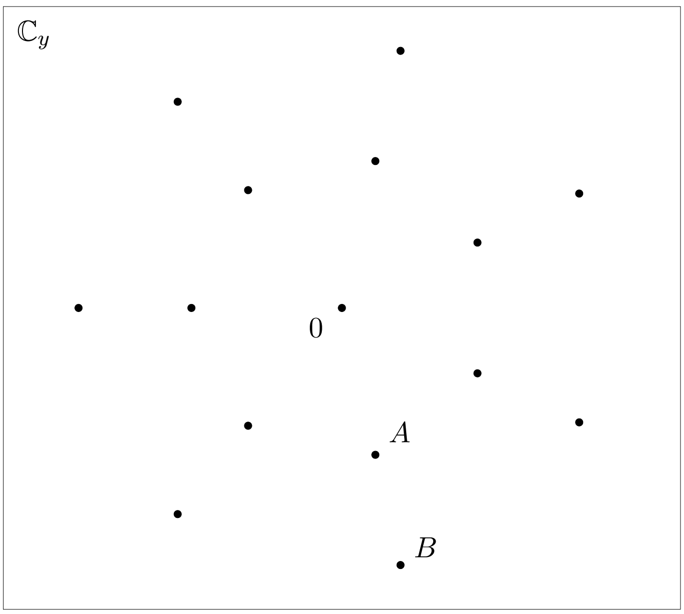
At $\alpha = 0$ we have

$$F_g|_{\alpha=0} \sim x^3 + xy^3 + z^2,$$

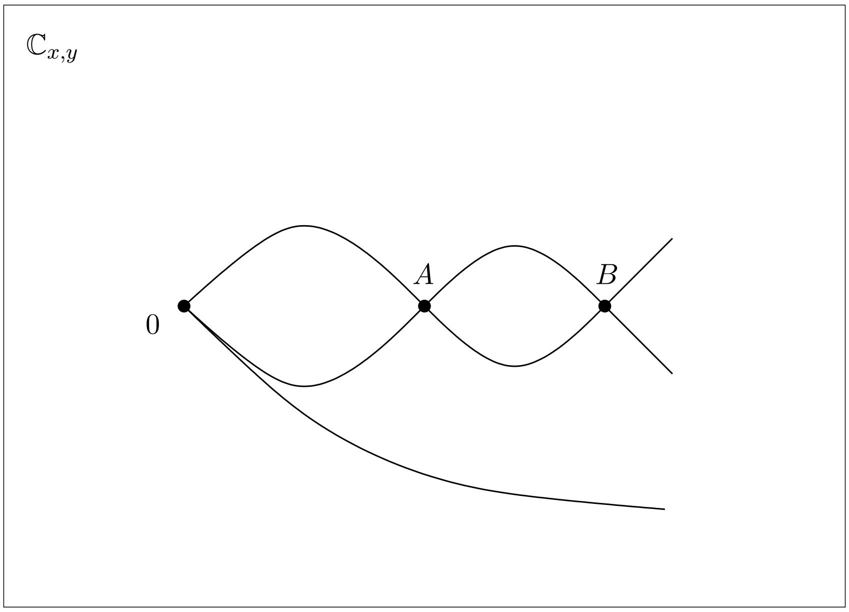
a critical point of type E_7. For $\alpha \neq 0$ (i.e. $y \neq 0$) we find the relation $\alpha = (-\beta/3)^3$, meaning the discriminant is of type G_2. Since a generic line intersects this component once, and singularities corresponding to this component have multiplicity 7, they must be of type $7A_1$ to satisfy $\Sigma\mu_i \frac{w_{f_i}}{w_{\alpha_i}} = \mu \frac{w_f}{w_\alpha}$. In two variables, for generic value of α, β, consider the zero level of the function of a branched cover of $\mathbb{C}_y$. The branching points are zeros of Σ_x, which has equation

$$\begin{aligned} \Sigma_x &= (\alpha y + y^8)^2 + (\beta y^3)^3 \\ &= y^2(y^{14} + (2\alpha + \beta^3)y^7 + \alpha^2) = 0. \end{aligned}$$

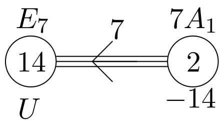
Choosing α, β so that the values of y^7 are real negative, we have the following picture in $\mathbb{C}_y$ for the branching points. The branching at $y = 0$ is of order 3, elsewhere of order 2.



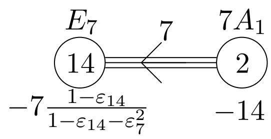
This picture is for a base point $\star \in \Lambda \setminus \Sigma$. As $\star$ moves to the component corresponding to the E_7 singularity, the interior points collapse to the origin. As $\star$ moves to the component corresponding to the $7A_1$ singularities, we collapse the pairs of order 2 branching points. Joining the inner branching points along the involved pairs of sheets to the origin of the curve, we get the 7 circles from which we make the vanishing E_7 χ'-cycle.



Therefore, in the two variable case the intersection of the two χ-cycles in 7 and this survives adding z^2. We begin to construct the Dynkin diagram.



We have still to find the self-intersection number of the cycle corresponding to E_7. Instead of doing this directly, we use the fact that the generators satisfy the relation $(h_1h_2)^3 = (h_2h_1)^3$. The unknown U is the solution of a linear equation and thus the solution is unique.



4.3.6 $Z_{12} \ni x^3y + xy^4 + z^2$

The Coxeter element and basic equivariants are

$$\begin{aligned} C(x,y,z) &= (\varepsilon_{11}^3 x, \varepsilon_{11}^2 y, -z) \\ g_x(x,y,z;f) &= (\varepsilon_8^3 x, iy, \varepsilon_{16}^3 z; \varepsilon_8^3 f) \\ g_y(x,y,z;f) &= (\omega x, \varepsilon_9^2 y, \varepsilon_{18}^{11} z; \varepsilon_9^2 f). \end{aligned}$$

We have $g_x = \iota_z C^{\frac{11}{8}}$, $g_y = C^{\frac{11}{9}}$. Since the determinants of the matrices of exponents are equal to orders of the respective basic equivariants, all symmetries are powers of these basic equivariants.

f	$g =$	$\lvert g\rvert$	versal monomials	notation
$x^3y + xy^4 + z^2 \in Z_{12}$	g_x	16	x	$-$
	g_y, g_y^2	$18, 9$	y	$-$
	g_x^2	8	x, xy^2	$Z_{12}/\mathbb{Z}_8$
	g_y^3, g_y^6	$6, 3$	y, xy, y^4	$Z_{12}/\mathbb{Z}_6$
	g_x^4	4	x, xy, xy^2, x^3	$Z_{12}/\mathbb{Z}_4$

The singularities $Z_{12}/\mathbb{Z}_8$ and $Z_{12}/\mathbb{Z}_4$ are not smoothable since in both cases the deformation is of the form

$$F_g = x\psi(x,y) + z^2.$$

The singularity $Z_{12}/\mathbb{Z}_6$ is not smoothable since its deformation is of the form

$$F_g = y\psi(x,y) + z^2.$$

4.3.7 $W_{12} \ni x^4 + y^5 + z^2$

The Coxeter element and basic equivariants are given by

$$\begin{aligned} C(x,y,z) &= (ix, \varepsilon_5 y, -z) \\ g_x(x,y,z;f) &= (\omega x, \varepsilon_{15}^4 y, \varepsilon_6 z; \omega f) \\ g_y(x,y,z;f) &= (\varepsilon_{16}^5 x, iy, \varepsilon_8^5 z; if). \end{aligned}$$

We have $g_x = \iota_z C^{\frac{4}{3}}$, $g_y = C^{\frac{5}{4}}$. The determinants of exponents satisfy $\Delta_x = 30 = |g_x|$, $\Delta_y = 32 = 2|g_y|$. The factor of 2 in $|g_y|$ may be ignored due to the usual stabilisation considerations, and we consider only symmetries arising as powers of the basic equivariants.

f	$g=$	$\|g\|$	versal monomials	notation
$x^4+y^5+z^2 \in W_{12}$	g_y	16	y	$-$
	g_x, g_x^2	$30, 15$	x	$-$
	g_y^2	8	y, y^3	$W_{12}/\mathbb{Z}_8$
	g_x^5, g_x^{10}	$6, 3$	x, y^2, x^2y, xy^3	$W_{12}/\mathbb{Z}_6$

<u>$W_{12}/\mathbb{Z}_8$</u>

The singularity $W_{12}/\mathbb{Z}_8$ has deformation and partial derivatives

$$\begin{aligned} F_g &= x^4+y^5+z^2+\beta y^3+\alpha y \\ F_{g,x} &= 4x^3 \\ F_{g,y} &= 5y^4+3\beta y^2+\alpha. \end{aligned}$$

At any critical point we must have $x=0$, so by considering the variable y we see the discriminant is of type B_2. At $\alpha = 0$ we have

$$F_g|_{\alpha=0} \sim x^4+y^3+z^2,$$

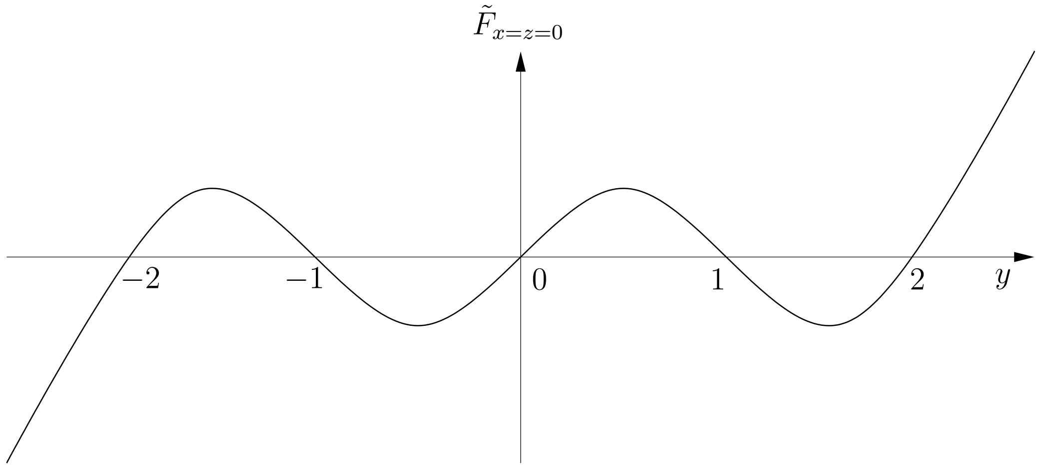
a critical point of type E_6.

Since a generic line intersects the $\alpha \neq 0$ component once, and singularities corresponding to this component have multiplicity 2, they must be of type $2A_3$ to satisfy $\Sigma \mu_i \frac{w_{f_i}}{w_{\alpha_i}} = \mu \frac{w_f}{w_\alpha}$.

To find the intersection number between the cycles we use similar methods to those given on Page 79 for $E_{12}/\mathbb{Z}_9$. Consider the deformation

$$\tilde{F}_{x=z=0} = y(y^2-1)(y^2-4),$$

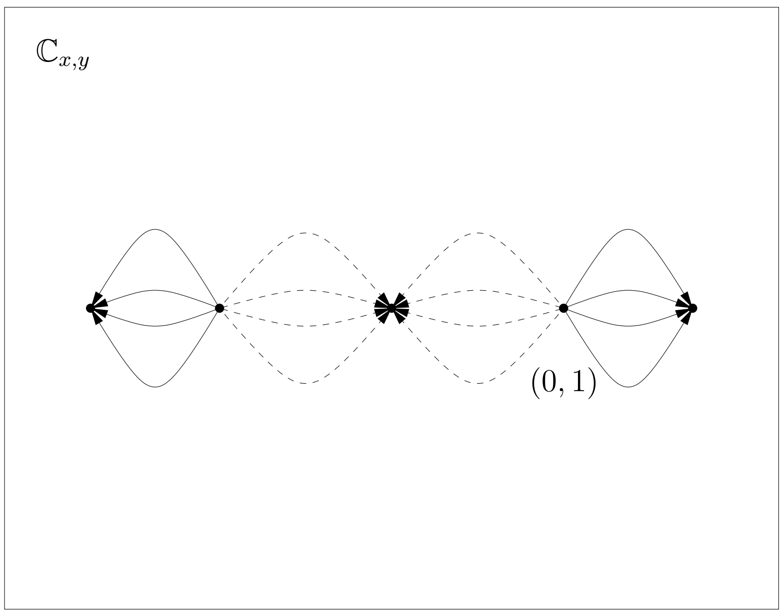
and its graph.



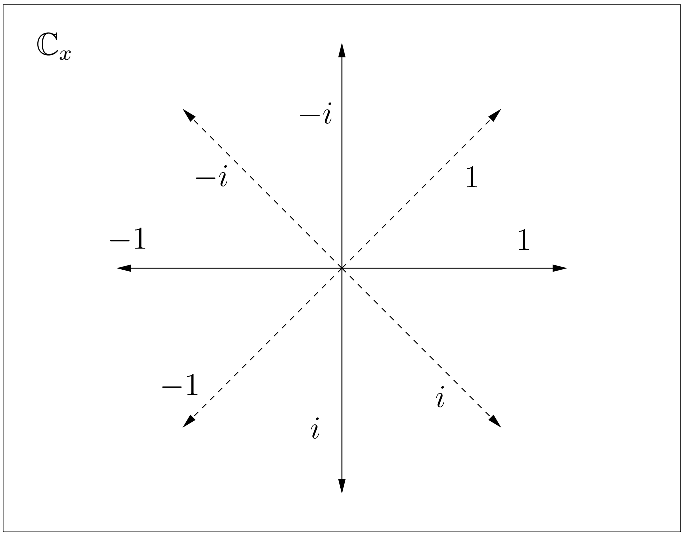
Adding the monomial x^4 we get

$$\tilde{F}_{z=0} = x^4 + y(y^2-1)(y^2-4).$$

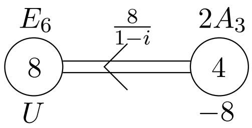
The branching points giving by the zero level of this surface are roots of $\tilde{F}_{x=z=0}$. The schematic picture is as follows.



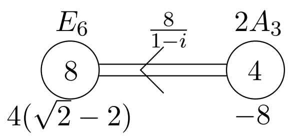
To find the intersection number of the two cycles, consider the picture at $(x,y)=(0,1)$ with local coordinate x.



So the local intersection number in 2 variables is the same as the intersection number between two cycles of type A_3, and this is given in Section 3.2. This number survives the addition of the third variable z^2 under the usual stabilisation procedure, and since we must also consider the point $(x, y) = (0, -1)$, we take twice this number giving the Dynkin diagram as follows.



We have still to find the self-intersection number of the cycle corresponding to E_6. Instead of doing this directly, we use the fact that the generators satisfy the relation $(h_1 h_2)^2 = (h_2 h_1)^2$. The unknown U is the solution of a linear equation and thus the solution is unique. Note that the self-intersection number of the cycle corresponding to the E_6 singularity is different to what has been seen earlier due to a different symmetry (that is $E_{12}/\mathbb{Z}_9$ on page 79).



$\underline{W_{12}/\mathbb{Z}_6}$

The singularity $W_{12} \ni x^4 + y^5 + z^2$ has quasidegree 20, the variables have weights 5, 4 and 10 respectively. Consider the deformation

$$\begin{aligned} F_g &= x^4 + y^5 + z^2 + \delta x y^3 + \gamma x^2 y + \beta y^2 + \alpha x \\ F_{g,x} &= 4x^3 + \delta y^3 + 2\gamma x y + \alpha \\ F_{g,y} &= 5y^4 + 3\delta x y^2 + \gamma x^2 + 2\beta y. \end{aligned}$$

The ratio of the weights of the parameters

$$\delta : \gamma : \beta : \alpha = 1 : 2 : 4 : 5$$

is not seen anywhere in the Shephard-Todd classification, and so our discriminant is of unknown type.

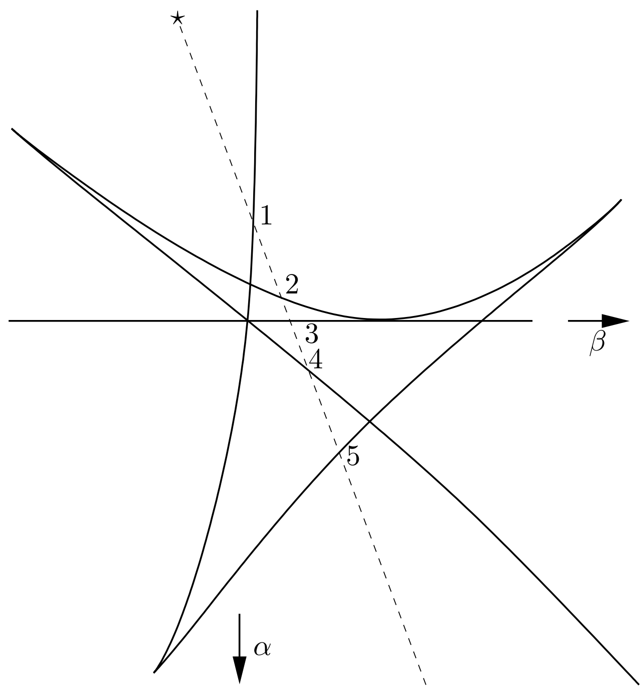
We note the adjacency $W_{12}/\mathbb{Z}_6 \to X_9/\mathbb{Z}_6$, and so our diagram is obtained by adding a vertex to the known diagram for $X_9/\mathbb{Z}_6$, which is given in Section 3.4.2.

If we assume $xy = 0$ in F_g and its derivatives, we find that $x = y = 0$. The discriminant component corresponding to such critical points of F_g is $\Sigma_1 = \{\alpha = 0\}$, and F_g has singularities of type A_3 on this component. Next assume $xy \neq 0$. The discriminant component in this case is

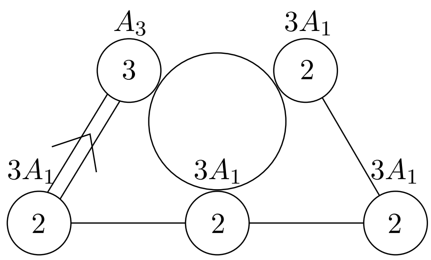
$$\begin{aligned} \Sigma_2 = \{ \quad & -8\beta^2\gamma^5\delta^2 - 5000\delta\beta\alpha^3 + \beta^2\gamma^4\delta^4 - 8\gamma^2\beta^3\delta^4 - 27\delta^6\alpha^2\beta - 128\beta^4\delta^2\gamma + \\ & +64\gamma^3\beta^3\delta^2 + 500\alpha^3\delta\gamma^2 + 4000\gamma\beta^2\alpha^2 - 225\alpha^3\delta^3\gamma + 1800\delta^2\beta^2\alpha^2 + \\ & +32\delta^3\beta^3\alpha - 200\beta\gamma^3\alpha^2 + \alpha^2\gamma^3\delta^4 - 8\alpha^2\gamma^4\delta^2 + 768\gamma^2\beta^4 + 16\gamma^5\alpha^2 + \\ & +16\beta^2\gamma^6 - 192\beta^3\gamma^4 + 16\beta^4\delta^4 + 27\alpha^3\delta^5 - \delta^5\gamma^3\alpha\beta + 8\delta^3\gamma^4\alpha\beta - \\ & -16\delta\gamma^5\alpha\beta + 704\delta\gamma^3\alpha\beta^2 - 296\delta^3\gamma^2\alpha\beta^2 - 2560\delta\gamma\alpha\beta^3 + 36\delta^5\gamma\alpha\beta^2 + \\ & +216\delta^4\alpha^2\beta\gamma - 430\gamma^2\alpha^2\delta^2\beta + 3125\alpha^4 - 1024\beta^5 = 0 \quad \}. \end{aligned}$$

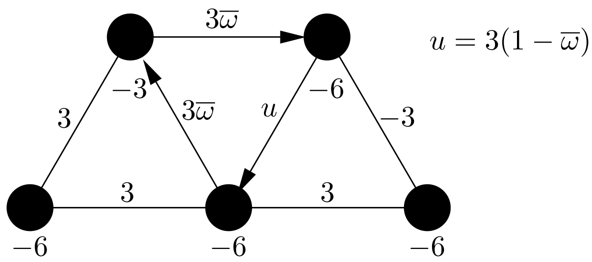
A generic section is found by setting γ, δ sufficiently large and negative. The diagram is given below along with a generic line. In this section the component Σ_2 is isomorphic to a generic section of an A_4 discriminant, the

Σ_1 component is a line arranged in such a way as to create a triple point and a tangential point of the generic section. We number the loops around the discriminant according to the order in which they leave the point $\star$ in the anticlockwise direction.



The majority of relations come from the adjacency to $X_9/\mathbb{Z}_6$. In our notation the new generator is h_5. In the diagram, h_5 corresponds to the lower right hand vertex. Travelling clockwise, other vertices correspond to h_4, h_2, h_3, h_1 respectively.





Most intersection numbers come from the adjacency, the remaining from Section 3.2. The weights of the deformation parameters are in the ratio $(1:2:4:5)$. This has not been seen before as a ratio weights of basic invariants of groups in any classification, and the skeleton is new.

The χ'-cycles corresponding to the generators connected by the curved edges satisfy the relation $e_3 = e_4 + e_5$.

4.3.8 $Q_{11} \ni x^2z + y^3 + yz^3$

The Coxeter element and basic equivariants are

$$\begin{aligned} C(x,y,z) &= (\varepsilon_{18}^7 x, \omega y, \varepsilon_9^2 z) \\ g_x(x,y,z;f) &= (\varepsilon_{11}^7 x, \varepsilon_{11}^6 y, \varepsilon_{11}^4 z; \varepsilon_{11}^7 f) \\ g_y(x,y,z;f) &= (\varepsilon_{12}^7 x, -y, \omega z; -f) \\ g_z(x,y,z;f) &= (-x, \varepsilon_7^3 y, \varepsilon_7^2 z; \varepsilon_7^2 f). \end{aligned}$$

We have $g_x = C^{\frac{18}{11}}$, $g_y = C^{\frac{3}{2}}$ and $g_z = C^{\frac{9}{7}}$. Since the determinants of the matrices of exponents are equal to the orders of the basic equivariants, powers of the basic equivariants give all symmetries.

f	$g =$	$\lvert g\rvert$	versal monomials	notation
$x^2z + y^3 + yz^3 \in Q_{11}$	g_z, g_z^2	$14, 7$	z	$Q_{11}/\mathbb{Z}_{14}$
	g_y	12	y	$Q_{11}/\mathbb{Z}_{12}$
	g_x	11	x	$Q_{11}/\mathbb{Z}_{11}$
	g_y^3	4	y, yz, yz^2	$Q_{11}/\mathbb{Z}_4$

The singularity $Q_{11}/\mathbb{Z}_4$ has deformation

$$F_g = x^2z + y^3 + yz^3 + \gamma yz^2 + \beta yz + \alpha y.$$

This is not smoothable since it is of the form

$$F_g = x^2z + y\psi(y, z, \lambda)$$

and has a critical point whenever

$$x = y = \psi(0, z, \lambda) = 0.$$

4.3.9 $Z_{13} \ni x^3y + y^6 + z^2$

The Coxeter element and basic equivariants are given by

$$\begin{aligned} C(x, y, z) &= (\varepsilon_{18}^5 x, \varepsilon_6 y, -z) \\ g_x(x, y, z; f) &= (\varepsilon_{13}^5 x, \varepsilon_{13}^3 y, \varepsilon_{26}^5 z; \varepsilon_{13}^5 f) \\ g_y(x, y, z; f) &= (\omega x, \varepsilon_5 y, \varepsilon_{10} z; \varepsilon_5 f) \end{aligned}$$

We have $g_x = \iota_z C^{\frac{18}{13}}$ and $g_y = \iota_z C^{\frac{6}{5}}$. The determinant $\Delta_x = 13 = |g_x|$ and so all symmetries preserving the monomial x are powers of g_x. We find further that $\Delta_y = 30 = |g_y|$, so all symmetries preserving the monomial y are powers of g_y.

f	$g =$	$\lvert g\rvert$	versal monomials	notation
$x^3y + y^6 + z^2 \in Z_{13}$	g_y, g_y^2	$30, 15$	y	–
	g_x, g_x^2	$26, 13$	x	–
	g_y^3, g_y^5	$10, 6$	y, xy	$Z_{13}/\mathbb{Z}_{10}$

The singularity $Z_{13}/\mathbb{Z}_{10}$ is not smoothable since its deformation is of the form

$$F_g = y\psi(x, y, \lambda) + z^2.$$

4.3.10 $S_{11} \ni x^2z + yz^2 + y^4$

The Coxeter element and basic equivariants are given by

$$\begin{aligned} C(x,y,z) &= (\varepsilon_{16}^5 x, iy, \varepsilon_8^3 z) \\ g_x(x,y,z;f) &= (\varepsilon_{11}^5 x, \varepsilon_{11}^4 y, \varepsilon_{11}^6 z; \varepsilon_{11}^5 f) \\ g_y(x,y,z;f) &= (\varepsilon_{12}^5 x, \omega y, -z; \omega f) \\ g_z(x,y,z;f) &= (-x, \varepsilon_5^2 y, \varepsilon_5^3 z; \varepsilon_5^3 f). \end{aligned}$$

We have $g_x = C^{\frac{16}{11}}$, $g_y = C^{\frac{4}{3}}$ and $g_z = C^{\frac{8}{5}}$. The orders of the matrices of exponents are equal to the orders of the basic equivariants, and so symmetries are all powers of the basic equivariants.

f	$g =$	$\lvert g\rvert$	versal monomials	notation
$x^2z + yz^2 + y^4 \in S_{11}$	g_y	12	y	–
	g_x	11	x	–
	g_z, g_z^2	$10, 5$	z	–
	g_y^2	6	y, yz	$S_{11}/\mathbb{Z}_6$
	g_y^4	3	y, yz, xy^2	$S_{11}/\mathbb{Z}_3$

The singularities $S_{11}/\mathbb{Z}_6$ and $S_{11}/\mathbb{Z}_3$ are not smoothable since in both cases the deformation is of the form

$$F_g = y\psi(x,y,z,\lambda) + x^2 z.$$

4.3.11 $W_{13} \ni x^4 + xy^4 + z^2$

The Coxeter element and basic equivariants are given by

$$\begin{aligned} C(x,y,z) &= (ix, \varepsilon_{16}^3 y, -z) \\ g_x(x,y,z;f) &= (\omega x, iy, \varepsilon_6 z; \omega f) \\ g_y(x,y,z;f) &= (\varepsilon_{13}^4 x, \varepsilon_{13}^3 y, \varepsilon_{13}^8 z; \varepsilon_{13}^3 f). \end{aligned}$$

We have $g_x = \iota_z C^{\frac{4}{3}}$, $g_y = C^{\frac{16}{13}}$. Again we observe the determinants of the exponent matrices are the same as the orders of the basic equivariants except for a factor of 2 for g_x coming from the z coordinate, which can be ignored since the singularity is stably equivalent to a function of two variables.

f	$g =$	$\lvert g\rvert$	versal monomials	notation
$x^4 + xy^4 + z^2 \in W_{13}$	g_y	13	y	$-$
	g_x	12	x	$-$
	g_x^2	6	x, xy^2	$W_{13}/\mathbb{Z}_6$
	g_x^4	3	x, xy, xy^2	$W_{13}/\mathbb{Z}_3$

The singularities $W_{13}/\mathbb{Z}_6$ and $W_{13}/\mathbb{Z}_3$ are not smoothable since in both cases the deformation is of the form

$$F_g = x\psi(x, y) + z^2.$$

4.3.12 $Q_{12} \ni x^2z + y^3 + z^5$

The Coxeter element and basic equivariants are

$$\begin{aligned}
C(x, y, z) &= (\varepsilon_5^2 x, \omega y, \varepsilon_5 z) \\
g_x(x, y, z; f) &= (\varepsilon_6 x, \varepsilon_{18} y, \varepsilon_6^5 z; \varepsilon_6 f) \\
g_y(x, y, z; f) &= (\varepsilon_{10} x, -y, \varepsilon_{10}^3 z; -f) \\
g_z(x, y, z; f) &= (-x, \varepsilon_{12}^5 y, iz; if).
\end{aligned}$$

We have $g_x = \iota_{x,y,z} C^{\frac{5}{3}}$, $g_y = \iota_x C^{\frac{3}{2}}$, $g_z = C^{\frac{5}{4}}$, where

$$\iota_{x,y,z}(x, y, z) = (-x, -y, -z).$$

So all symmetries preserving the monomial x are powers of g_x.

The determinant of the matrix of exponents coincides with the order of the basic equivariant in the case $\Delta_x = 18 = |g_x|$. For g_y and g_z we have

$$\Delta_y = 20 = 2|g_y|,$$

and

$$\Delta_z = 24 = 2|g_z|.$$

So all symmetries preserving the monomial y (resp. z) are given by powers of g_y (resp. g_z) and compositions of these powers with the involution $\iota_x(x,y,z) = (-x,y,z)$, corresponding to the symmetry of the Dynkin diagram of Q_{12} (shown in Figure 2.2.1 on Page 13).

f	$g =$	$\lvert g\rvert$	versal monomials	notation
$x^2z + y^3 + z^5 \in Q_{12}$	g_x	18	x	$-$
	$g_z, \iota_x g_z$	12	z	$-$
	$g_y, \iota_x g_y$	10	y	$-$
	g_x^2	9	x, z^2	$Q_{12}/\mathbb{Z}_9$
	$g_x^3, \iota_x g_z^2$	6	z, x, z^3	$Q_{12}/\mathbb{Z}_6$
	g_z^2	6	z, z^3	$(Q_{12}/\mathbb{Z}_6)'$
	g_z^3	4	z, xy, yz^2	$Q_{12}/\mathbb{Z}_4$
	$\iota_x g_z^3$	4	z, yz^2	$(Q_{12}/\mathbb{Z}_4)'$
	g_x^9, g_y^5, g_z^6	2	z, y, x, z^3, z^2, yz^4	$Q_{12}/\mathbb{Z}_2$
	$\iota_x g_y^5, \iota_x g_z^6$	2	$z, y, z^3, xy, yz^2, yz^4$	$(Q_{12}/\mathbb{Z}_2)'$

<u>$Q_{12}/\mathbb{Z}_9$</u>

In this case we have

$$\begin{aligned}
F_g &= x^2z + y^3 + z^5 + \beta z^2 + \alpha x \\
F_{g,x} &= 2xz + \alpha \\
F_{g,y} &= 3y^2 \\
F_{g,z} &= x^2 + 5z^4 + 2\beta z.
\end{aligned}$$

So at any critical point $y = 0$. Assuming $z = 0$, then we have also that $x = 0$ giving the component $\alpha = 0$ for the singularity. If we assume $z \neq 0$, we find also that $x \neq 0$ and we get a second component for the discriminant given by the equation $\alpha^2 + \beta^2 = 0$. On the $\alpha = 0$ component we have

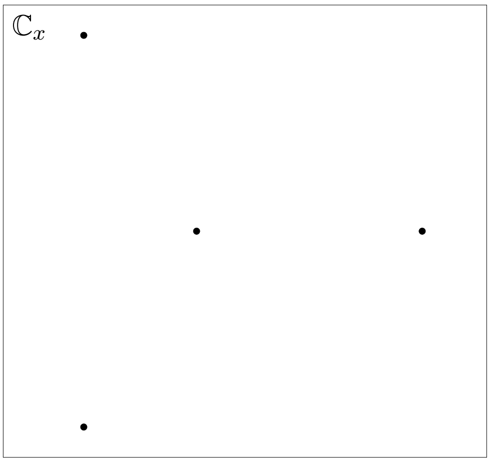
$$F_g|_{\alpha=0} \sim x^4 + y^3 + z^2,$$

a critical point of type E_6. Since a generic line intersects the $\alpha \neq 0$ component

twice, and singularities corresponding to this component have multiplicity 3, they must be of type $3A_2$ to satisfy $\Sigma\mu_i\frac{w_{f_i}}{w_{\alpha_i}} = \mu\frac{w_f}{w_\alpha}$.
We start by calculating the self-intersection of the cycles corresponding to the E_6 singularity. Define the deformation

$$\tilde{F}_{y=z=0} = x^4 - x,$$

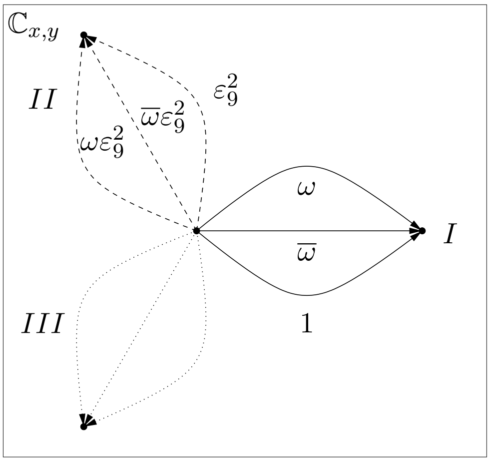
which has roots $x = 0, 1, \omega, \overline{\omega}$. We draw these in the $\mathbb{C}_x$ plane.



Adding the variable y to get the deformation

$$\tilde{F}_{z=0} = x^4 - x + y^3$$

gives an order 3 covering of the space branched at the roots of $\tilde{F}_{y=z=0}$. The schematic picture is given.

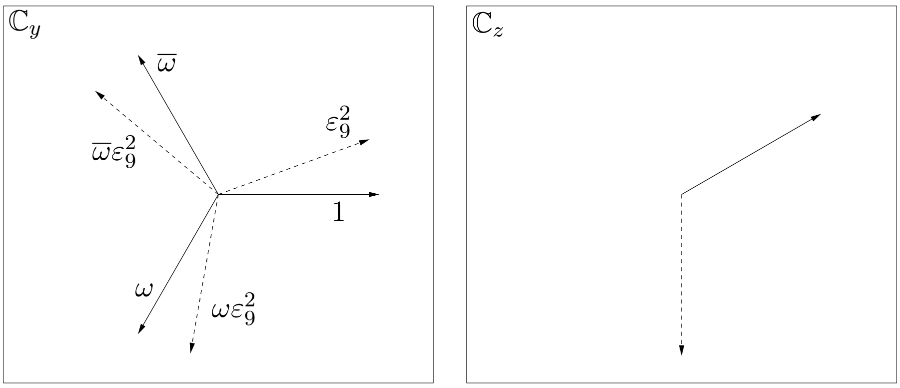


For simplicity we reduce the cycle into the union of three cycles, labelled I, II, III above. The labelling comes from the character

$$\chi' = \frac{\omega\varepsilon_9\overline{\omega}}{\omega} = \overline{\varepsilon}_9^2,$$

and continues for III in the cyclic way.

Consider just the intersection between cycle I and cycle II in 3 variables by including z such that $\tilde{F} = x^4 - x + y^3 + z^2$. The local picture is given in the $\mathbb{C}_y$ and $\mathbb{C}_z$ direction respectively.



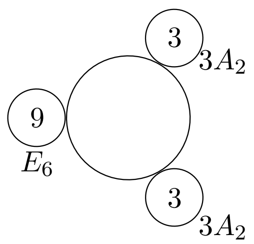
To calculate the intersection number $\langle I, II\rangle$ we make standard calculations in the $\mathbb{C}_y$ direction, remembering the result should be taken with -1 according to what happens in the $\mathbb{C}_z$ direction. We repeat this method for $\langle I, II\rangle$, and find

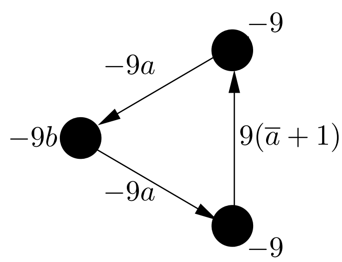
$$\begin{aligned}\langle I, II\rangle &= \overline{\varepsilon_9}^2\frac{3}{1-\omega}\\ \langle II, I\rangle &= \varepsilon_9^2\frac{3}{1-\overline{\omega}}.\end{aligned}$$

The self-intersection number $\langle I, I\rangle = -3$ using methods identical to the A_3 case given in Section 3.2. Adding these gives

$$\langle I, II\rangle + \langle II, I\rangle + \langle I, I\rangle = \overline{\varepsilon_9}^2\frac{3}{1-\omega} + \varepsilon_9^2\frac{3}{1-\overline{\omega}} - 3.$$

The total self-intersection number is 3 times this. Similar calculations to those done for $Q_{10}/\mathbb{Z}_5$ on page 85 give the other intersection numbers, and the Dynkin diagram is given.





$$\begin{aligned}a &= \frac{1-\overline{\varepsilon_9}^2}{\omega - 1}\\ b &= \frac{1}{(\varepsilon_9+\overline{\varepsilon_9})(\varepsilon_9+\overline{\varepsilon_9}+1)}\end{aligned}$$

If generators h_1, h_2, h_3 correspond to cycles e_1, e_2, e_2, the relation on these cycles is $e_1 + e_2 + e_3 = 0$.

$\underline{Q_{12}/\mathbb{Z}_6}$

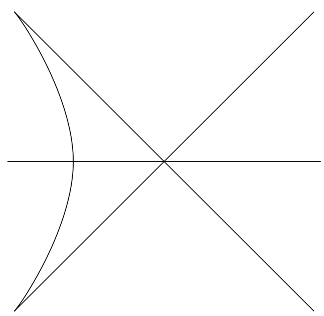
We have:

$$\begin{aligned} F_g &= x^2z + y^3 + z^5 + \gamma z^3 + \beta x + \alpha z \\ F_{g,x} &= 2xz + \beta \\ F_{g,y} &= 3y^2 \\ F_{g,z} &= x^2 + 5z^4 + 3\gamma z^2 + \alpha. \end{aligned}$$

So at any critical point we have $y = 0$. If $z = 0$, we get a component of the discriminant with $\beta = 0$. If $z > 0$ is fixed, we get a component with equation

$$27\beta^4 + 72\gamma\beta^2\alpha + 64\alpha^3 - 16\gamma^2\alpha^2 - 16\gamma^3\beta^2 = 0,$$

a swallowtail. So the discriminant is the union of a swallowtail with its plane of symmetry. A generic two dimensional section is given.

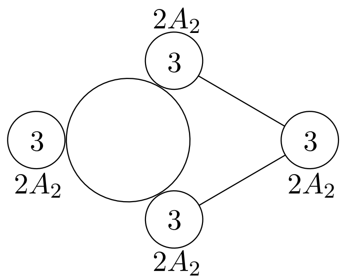


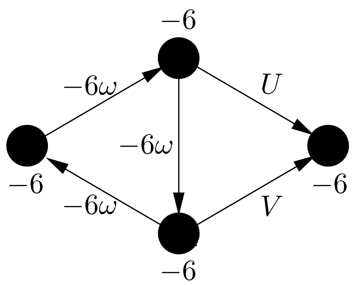
We notice that this singularity is adjacent to a known singularity

$$Q_{12}/\mathbb{Z}_6 \to P_8/\mathbb{Z}_6,$$

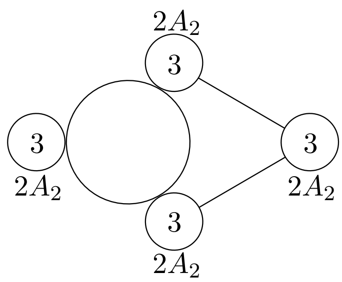
details of which can be found in Section 3.4.1. The discriminant in the P_8 case is just three intersecting lines, which can be seen near the triple point in the generic section of the discriminant of $Q_{12}/\mathbb{Z}_6$. All components of the discriminant in the P_8 case correspond to singularities of type $2A_2$, as they therefore must also do in the Q_{12} case.

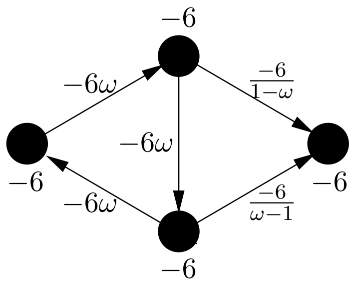
Information from the discriminant and from the known singularity may be combined to begin constructing the Dynkin diagram and intersection diagram for this singularity. The ratio of quasi-degrees of the parameters coincides with that of the Shephard-Todd group $G(6,2,3)$. Since braiding and braid-like relations in this group are also satisfied by our generators, we use the convention of [7] to construct our Dynkin diagram.





The known ternary relations coming from P_8 imply that $U = -V$. From Section 3.2 we may take $U = \frac{-6}{1-\omega}$.





We denote the leftmost of the three generators connected by curved edges by h_2, and travelling clockwise the other two are h_4, h_3 respectively to match Table 5.1 on page 127. The corresponding χ'-cycles satisfy $e_2 + e_3 + e_4 = 0$.

<u>$Q_{12}/\mathbb{Z}_4$</u>

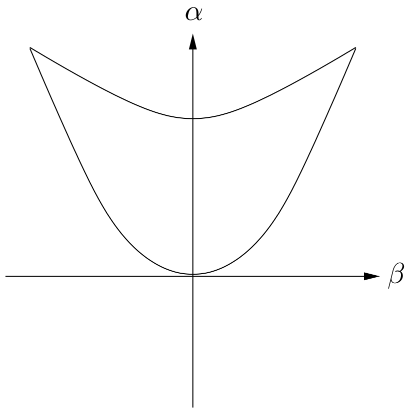
The singularity $Q_{12}/\mathbb{Z}_4$ has deformation and partial derivatives

$$\begin{aligned} F_g &= x^2z + y^3 + z^5 + \gamma yz^2 + \beta xy + \alpha z \\ F_{g,x} &= 2xz + \beta y \\ F_{g,y} &= 3y^2 + \gamma z^2 + \beta x \\ F_{g,z} &= x^2 + 5z^4 + 2\gamma yz + \alpha. \end{aligned}$$

Assume at a critical point we have $y = 0$. This implies that $x = 0$ or $z = 0$. Consider first the case $x = 0$. This further implies that $z = 0$, and defines a component of the discriminant given by the equation $\alpha = 0$. Next assume that $z = 0$ but $x \neq 0$. This corresponds to a component with equation $\beta = 0$. Finally, assume $y \neq 0$. The final component of the discriminant has equation

$$\begin{aligned} 108\alpha^2 + \beta^6 + 4\gamma^3\alpha + 2\gamma^2\beta^4 + 36\alpha\gamma\beta^2 + \gamma^4\beta^2 &= 0 \\ \text{or} \quad 3\left(6\alpha + \beta^2\gamma + \frac{1}{9}\gamma^3\right)^2 + \left(\beta^2 - \frac{1}{3}\gamma^2\right)^3 &= 0. \end{aligned}$$

A real generic section is shown for $\gamma < 0$.

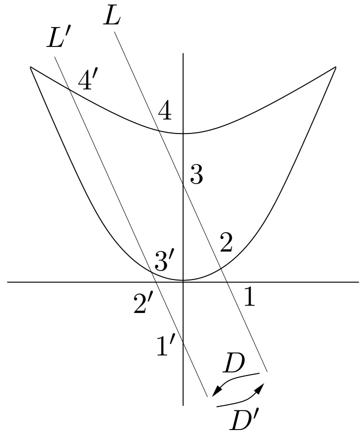


We observe using the adjacency

$$Q_{12}/\mathbb{Z}_4 \to P_8/\mathbb{Z}_4$$

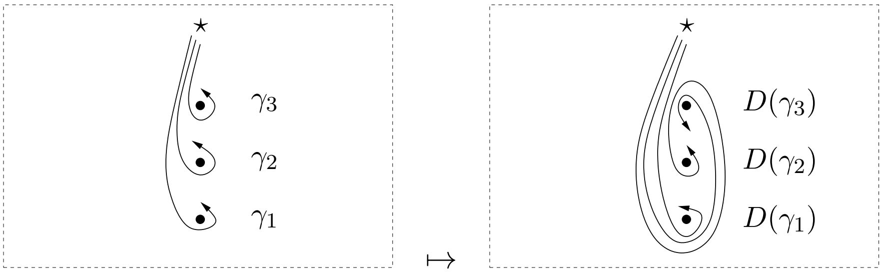
that the $\alpha = 0$ component of our discriminant corresponds to A_3 singularities, the $\beta = 0$ component to $2A_1$ singularities, and the final component to $4A_1$ singularities. A generic section of the discriminant in the P_8 case looks like the section of the Q_{12} discriminant (shown above) near the origin. However, this section is not maximal in the sense that some strata have gone complex.

Let us calculate the braiding relations for the generators of the fundamental group of the complement to the discriminant. We will take two generic lines in the complement to the discriminant, and the operations D, D' which drag them line continuously around the origin to each other, keeping track of both sets of generators. The action of $D' \circ D$ is homotopically equivalent to the identity action restricted to the generating loops.



Since it is clear from the picture that γ_4 and γ_4' do not interact with the other generators, we must have the equality $\gamma_4 = \gamma_4'$, and this generator is omitted through the following calculation.

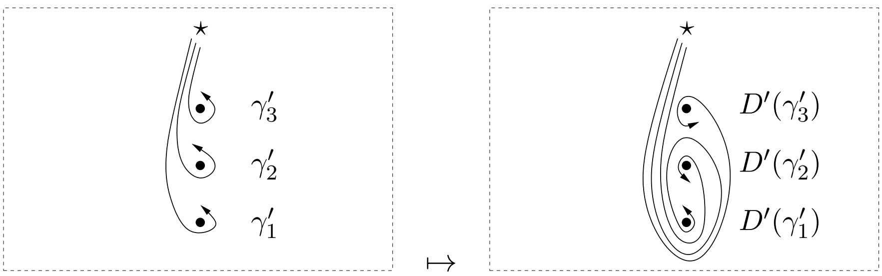
The operator D acts on the loops $\gamma_1, \gamma_2, \gamma_3$ in L, by sending them to loops in L' which may be expressed in terms of $\gamma_1', \gamma_2', \gamma_3'$.



The image of the loops under D are as follows.

$$
\begin{array}{lll}
D : & & \\
\gamma_1 & \mapsto & \bar{\gamma}_1'\bar{\gamma}_2'\bar{\gamma}_3'\gamma_2'\gamma_3'\gamma_2'\gamma_1' \\
\gamma_2 & \mapsto & \bar{\gamma}_1'\bar{\gamma}_2'\gamma_3'\gamma_2'\gamma_1' \\
\gamma_3 & \mapsto & \gamma_1'.
\end{array}
$$

The operator D' acts on the loops $\gamma_1', \gamma_2', \gamma_3'$ in L', by sending them to loops in L which may be expressed in terms of $\gamma_1, \gamma_2, \gamma_3$.



The image of the loops under D' are as follows.

$$\begin{array}{rcl} D': & & \\ \gamma_1' & \mapsto & \bar{\gamma}_1\bar{\gamma}_2\gamma_3\gamma_2\gamma_1 \\ \gamma_2' & \mapsto & \bar{\gamma}_1\bar{\gamma}_2\gamma_1\gamma_2\gamma_1 \\ \gamma_3' & \mapsto & \bar{\gamma}_1\gamma_2\gamma_1. \end{array}$$

We take the composition,

$$\begin{array}{rcl} D' \circ D: & & \\ \gamma_1 & \mapsto & \bar{\gamma}_1\bar{\gamma}_2\bar{\gamma}_3\bar{\gamma}_1\bar{\gamma}_2\gamma_1\gamma_2\gamma_1\gamma_3\gamma_2\gamma_1 \\ \gamma_2 & \mapsto & \bar{\gamma}_1\bar{\gamma}_2\bar{\gamma}_3\bar{\gamma}_1\gamma_2\gamma_1\gamma_3\gamma_2\gamma_1 \\ \gamma_3 & \mapsto & \bar{\gamma}_1\bar{\gamma}_2\gamma_3\gamma_2\gamma_1. \end{array}$$

Using the fact that $D' \circ D$ is homotopically equivalent to the identity on these loops, we deduce the relations,

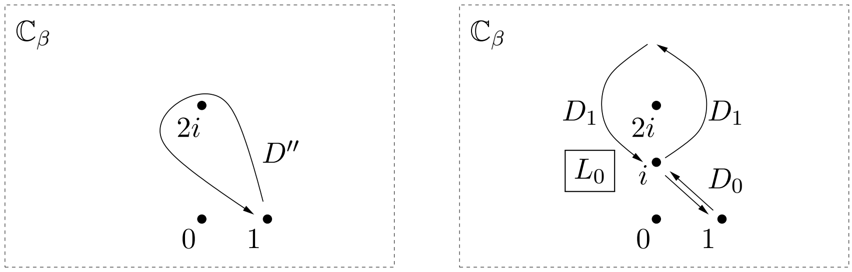
$$\begin{array}{rcl} \gamma_2\gamma_1\gamma_3\gamma_2\gamma_1 & = & \gamma_1\gamma_2\gamma_1\gamma_3\gamma_2 \\ \gamma_1\gamma_3\gamma_2\gamma_1\gamma_2 & = & \gamma_2\gamma_1\gamma_3\gamma_2\gamma_1 \\ \gamma_2\gamma_1\gamma_3 & = & \gamma_3\gamma_2\gamma_1, \end{array}$$

where the first relation is redundant as it is implied by the following two relations. By similar methods, we can also deduce the following relations.

$$\begin{array}{rcl} \gamma_4\gamma_3 & = & \gamma_3\gamma_4 \\ \gamma_2\gamma_4\gamma_2 & = & \gamma_4\gamma_2\gamma_4 \\ \gamma_2'\gamma_4\gamma_2' & = & \gamma_4\gamma_2'\gamma_4, \end{array}$$

where $\gamma_2' = \overline{\gamma_1}\gamma_2\gamma_1$.

We now consider the imaginary strata. Take for example the generic section with $\gamma = -2$. Then the point we are interested in is $(\alpha, \beta) = (0, 2i)$. We will drag a generic line isomorphic to $\mathbb{C}_\alpha$ around this point by the operator D'' shown below.



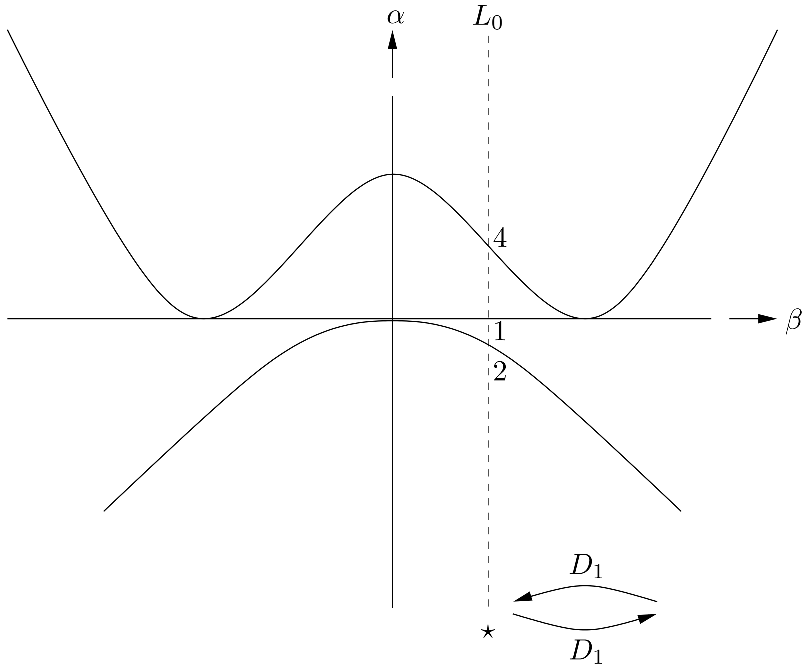
The operator D'' can be constructed in the following way

$$D'' = D_0^{-1} \circ D_1^2 \circ D_0.$$

The parameter β occurs only in even powers in the non-trivial component of the discriminant. This allows us to draw a real picture of the graph in $\mathbb{C}_{\alpha,\beta}$ by taking $\alpha \in \mathbb{R}, \beta \in i\mathbb{R}$. This is equivalent to sending $\beta \mapsto i\beta$ and considering the real graph. We get

$$108\alpha^2 - \beta^6 + 4\gamma^3\alpha + 2\gamma^2\beta^4 - 36\alpha\gamma\beta^2 - \gamma^4\beta^2 = 0,$$

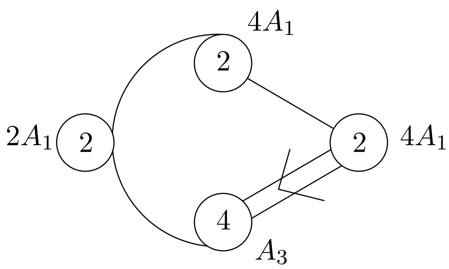
the graph of which is as follows.

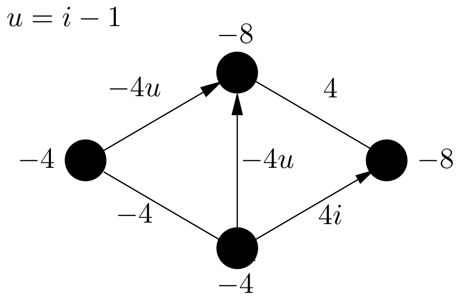


Taking L_0 to be the line (α, i), the operator D_1^2 gives the same results as when considering a B_2 discriminant, and we get the additional relation:

$$(\gamma_1\gamma_4)^2 = (\gamma_4\gamma_1)^2.$$

Since the relations on generators $\gamma_1, \gamma_2, \gamma_3$ are present in [7], e.g. for the lifted braid group corresponding to the Shephard-Todd group G_{15}, we adopt this Dynkin diagram as the subdiagram corresponding to these generators. In the diagram, the generator on the far right is h_4. Travelling around clockwise from here, the generators are h_4, h_1, h_3, h_2. Standard considerations respecting the known relations allow us to calculate the remaining intersection numbers.





The χ'-cycles e_1, e_2, e_3 corresponding to generators h_1, h_2, h_3 satisfy the relation $(1+i)e_2 = e_1 + e_3$.

<u>Non-smoothable deformations</u>

The singularities $(Q_{12}/\mathbb{Z}_6)'$ and $(Q_{12}/\mathbb{Z}_4)'$ are not smoothable since in both cases the deformation is of the form

$$F_g = z\psi(x, y, z, \lambda) + y^3.$$

<u>$Q_{12}/\mathbb{Z}_2$ and $(Q_{12}/\mathbb{Z}_2)'$</u>

In both cases the positive subspace in the cohomology is not split by the group. Indeed, the singularities are adjacent to similarly Z_2-equivariant P_8 functions whose $H_2^{\chi=1}$ (respectively $H_2^{\chi=-1}$) contains the whole rank 2 kernel of the P_8 intersection form (this follows from consideration of the cubes of the order 6 symmetries in lines 9 and 4 of Table 2 in [14]). Hence $H_2^{\chi=1}$ of $Q_{12}/\mathbb{Z}_2$ (respectively $H_2^{\chi=-1}$ of $(Q_{12}/\mathbb{Z}_2)'$) contains the whole rank 2 positive subspace of the Q_{12} intersection form.

We remark that in both cases the modular monomial yz^4 enters $\mathbb{Z}_2$-equivariant versal deformations.

4.3.13 $S_{12} \ni x^2z + yz^2 + xy^3$

The Coxeter element and basic equivariants are given by

$$\begin{aligned} C(x,y,z) &= (\varepsilon_{13}^4 x, \varepsilon_{13}^3 y, \varepsilon_{13}^5 z) \\ g_x(x,y,z;f) &= (\varepsilon_9^4 x, \omega y, \varepsilon_9^5 z; \varepsilon_9^4 f) \\ g_y(x,y,z;f) &= (\varepsilon_5^2, \varepsilon_{10}^3, -z; \varepsilon_{10}^3 f) \\ g_z(x,y,z;f) &= (-x, \varepsilon_8^3 y, \varepsilon_8^5 z; \varepsilon_8^5 f). \end{aligned}$$

We have $g_x = C^{\frac{13}{9}}$, $g_y = C^{\frac{13}{10}}$, $g_z = C^{\frac{13}{8}}$. Orders of matrices of exponents are equal to the order of the basic equivariants, so symmetries are powers of the basic equivariants.

f	$g =$	$\lvert g\rvert$	versal monomials	notation
$x^2z + yz^2 + xy^3 \in S_{12}$	g_y	10	y	–
	g_x	9	x	–
	g_z	8	z	–
	g_y^2	5	y, yz	$S_{12}/\mathbb{Z}_5$
	g_z^2	4	z, y^3	$S_{12}/\mathbb{Z}_4$
	g_x^3	3	x, xy^2, z^2	$S_{12}/\mathbb{Z}_3$
	g_y^5, g_z^4	2	y, z, xy, y^3, y^2z, y^5	$S_{12}/\mathbb{Z}_2$

The singularity $S_{12}/\mathbb{Z}_5$ has deformation

$$F_g = x^2z + yz^2 + xy^3 + \beta yz + \alpha y.$$

This is not smoothable since it has a zero level critical point whenever

$$z^2 + \beta z + \alpha = 0, x = y = 0.$$

The singularity $S_{12}/\mathbb{Z}_4$ has deformation

$$F_g = x^2z + yz^2 + xy^3 + \beta y^3 + \alpha z.$$

This is not smoothable since it has a zero level critical point whenever

$$x^2 + \alpha = 0, y = z = 0.$$

The singularity $S_{12}/\mathbb{Z}_3$ has deformation

$$F_g = x^2z + yz^2 + xy^3 + \gamma z^2 + \beta xy^2 + \alpha x.$$

This is not smoothable since it has a zero level critical point whenever

$$y^2 + \beta y^2 + \alpha = 0, x = z = 0.$$

$\underline{S_{12}/\mathbb{Z}_2}$

In this case the positive subspace in the cohomology is not split by the symmetry. Indeed, the singularity is adjacent to a similarly $\mathbb{Z}_2$-equivariant P_8 function whose $H_2^{\chi=-1}$ contains the whole rank 2 kernel of the P_8 intersection form (this follows from consideration of the cube of the order 6 symmetry given in line 4 of Table 2 in [14]). Therefore, $H_2^{\chi=-1}$ of $S_{12}/\mathbb{Z}_2$ contains the whole rank 2 positive subspace of the S_{12} intersection form.

We remark that the modular monomial y^5 enters $\mathbb{Z}_2$-equivariant versal deformation of $S_{12}/\mathbb{Z}_2$.

4.3.14 $U_{12} \ni x^3 + y^3 + z^4$

The Coxeter element and basic equivariants are

$$\begin{aligned} C(x,y,z) &= (\omega x, \omega y, iz) \\ g_x(x,y,z;f) &= (-x, \varepsilon_6^5 y, \varepsilon_8 z; -f) \\ g_y(x,y,z;f) &= (\varepsilon_6^5 x, -y, \varepsilon_8 z; -f) \\ g_z(x,y,z;f) &= (\varepsilon_9^4 x, \varepsilon_9^4 y, \omega z; \omega f). \end{aligned}$$

We have $g_x = \sigma C^{\frac{1}{2}}$, $g_y = \sigma^2 C^{\frac{1}{2}}$, $g_z = C^{\frac{4}{3}}$, recalling the map $\sigma(x,y,z) = (\omega x, \omega^2 y, z)$. The determinants of the matrices of exponents are

$$\begin{aligned} \Delta_x = 24 &= |g_x| \\ \Delta_y = 24 &= |g_y| \\ \Delta_z = 27 &= 3|g_z|. \end{aligned}$$

Therefore all symmetries preserving the monomials x and y are powers of g_x and g_y respectively. All symmetries preserving z are powers of g_z and powers of g_z composed with σ.

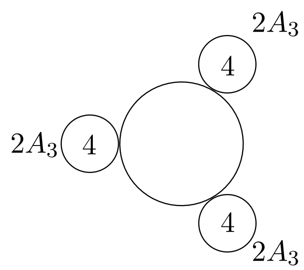
f	$g =$	$\lvert g\rvert$	versal monomials	notation
$x^3+y^3+z^4 \in U_{12}$	g_x	24	x	–
	g_y	24	y	–
	$g_z, \sigma g_z$	9	z	–
	g_x^3, g_y^3	8	x, y	$U_{12}/\mathbb{Z}_8$

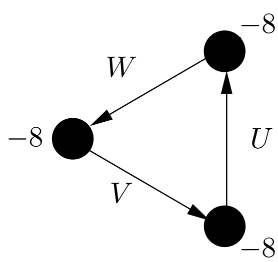
<u>$U_{12}/\mathbb{Z}_8$</u>

The singularity $U_{12}/\mathbb{Z}_8$ has deformation and partial derivatives

$$
\begin{aligned}
F_g &= x^3+y^3+z^4+\beta y+\alpha x \\
F_{g,x} &= 3x^2+\alpha \\
F_{g,y} &= 3y^2+\beta \\
F_{g,z} &= 4z^3,
\end{aligned}
$$

so at any critical point we have $z = 0$. Critical points occur whenever $\alpha = \omega^k\beta$, meaning the discriminant consists of three intersecting lines. A generic line intersects the discriminant at three points, each of these points corresponding to a singularity with orbit 2. Each component therefore corresponds to critical points of type $2A_3$ to satisfy $\Sigma\mu_i\frac{w_{f_i}}{w_{\alpha_i}} = \mu\frac{w_f}{w_\alpha}$. This can also be seen from the z^4 monomial in f which is not deformed by F_g.

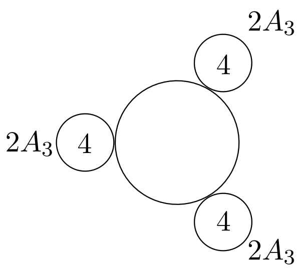


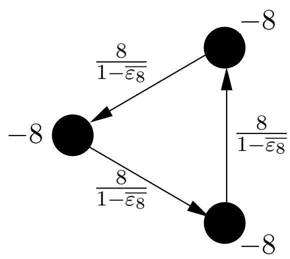


Take U, V, W above to be the unknown intersection numbers, and denote $\overline{U} = u$ and so on. We may assume the relation between the cycles is $e_1 + e_2 + Ke_3 = 0$, where e_1, e_2, e_3 are cycles corresponding to generators h_1, h_2, h_3 where h_1 is the leftmost generator and others are labelled anticlockwise in the diagram. Taking the intersection of this condition with each cycle in turn gives a system of simultaneous equations from which two unknowns may be eliminated, and which yields another condition.

The hyperbolicity implies $|U| > 8, |V| > 8, |W| > 8$, and we look for $U, V, W \in \mathbb{Z}\langle\varepsilon_8\rangle$ such that all relations are satisfied.

Writing generators explicitly in the chart e_2, e_3, we use similar methods to those used for $Q_{10}/\mathbb{Z}_5$ on page 85 to calculate the intersection numbers which are unique up to the usual ambiguities.





The χ'-cycles satisfy the relation $e_1 + e_2 + e_3 = 0$.

4.3.15 $U_{12} \ni x^2y + y^3 + z^4$

The Coxeter element and basic equivariants are given by

$$\begin{aligned} C(x,y,z) &= (\omega x, \omega y, iz) \\ g_x(x,y,z;f) &= (-ix, iy, \varepsilon_{16}^3 z; -if) \\ g_y(x,y,z;f) &= (-x, -y, \varepsilon_8^3 z; -f) \\ g_z(x,y,z;f) &= (\varepsilon_{18}^{17} x, \varepsilon_9^4 y, \omega x; \omega f). \end{aligned}$$

We have $g_x = \iota_x C^{\frac{3}{4}}$, $g_y = C^{\frac{3}{2}}$, $g_z = \iota_x C^{\frac{4}{3}}$. Also, $\Delta_x = 16 = |g_x|$ and $\Delta_z = 18 = |g_y|$. So all symmetries preserving the monomials x and z are powers of g_x and g_z respectively. We find $\Delta_y = 16 = 2|g_y|$, meaning symmetries preserving y are powers of g_y and powers of g_y composed with ι_x.

f	$g =$	$\lvert g\rvert$	versal monomials	notation
$x^2y + y^3 + z^4 \in U_{12}$	g_z, g_z^2	$18, 9$	z	$-$
	g_x	16	x	$-$
	g_x^2, g_y	8	x, y	$U_{12}/\mathbb{Z}_8$
	$\iota_x g_y$	8	y, xy	$(U_{12}/\mathbb{Z}_8)'$

The singularity $U_{12}/\mathbb{Z}_8$ is isomorphic to the one done in the previous section.

The singularity $(U_{12}/\mathbb{Z}_8)'$ is not smoothable since its deformation is of the form

$$F_g = y\psi(x,y,\lambda) + z^4,$$

and a similar argument holds.

This concludes the classification.

Chapter 5

Group presentations

The motivation for this chapter comes from the relationship discussed in [7] between generalised braid groups and Shepard Todd groups [21]. There, results are discussed regarding relations on the generators and details about the centre of the groups. In this chapter we give an exposition of experimental results due to the classification. These are similar in nature to those of [7] and are described in a similar way, but some things turn out harder to prove since in our case all the groups are infinite.

5.1 Derivation of presentations

We choose $\chi = \chi'$ so that our monodromy group has hyperbolic signature and recall that k is the number of basic elements from the local ring appearing in the g-versal deformation F_g. Then $M_{\chi'} \subset U(k-1,1)$ is a representation of the fundamental group of the complement to the discriminant. It is the image of the map

$$\rho_{\chi'} : \pi_1(\Lambda \setminus \Sigma, \star) \to GL(k, \mathbb{C}),$$

where as usual Λ and Σ respectively denote the base of a versal deformation and the discriminant. Following Zariski, choose a generic line and a generic plane containing this line in the base of our versal deformation: $L \subset P \subset \Lambda$.

For $\star \in L$ but $\star \notin \Sigma$ it holds that:

$$\pi_1(\Lambda \setminus \Sigma, \star) = \pi_1(P \setminus \{P \cap \Sigma\}, \star).$$

See for example [25]. We denote this group $\pi_1(\Sigma')$. Assume Σ has multiplicity ℓ. Then we generate $\pi_1(\Sigma')$ with ℓ elements: loops without mutual and self--intersections which travel from $\star$ around distinct points of $L \cap \Sigma$ (exactly as we defined them in Section 2.2.3 and onwards). Such a loop set is called *distinguished*. Denote them by $\gamma_1, \ldots, \gamma_\ell$ in the anticlockwise order as they leave $\star$. The relations between the generators may be read from $P \cap \Sigma$, and we denote this set of relations $\mathcal{B}(\{\gamma_i\})$. The group then has presentation:

$$\pi_1(\Sigma') = \langle \gamma_1, \ldots, \gamma_\ell | \mathcal{B}(\{\gamma_i\}) \rangle .$$

Remark 5.1. We observe from the classification that for invariant symmetries we have $k = \ell$, and for equivariant symmetries we have either $k = \ell$ or $k = \ell - 1$. In all cases $M_{\chi'}$ acts on a k dimensional space.

We denote the generators $\rho_{\chi'}(\gamma_i)$ of $M_{\chi'}$ by h_i. Since the monodromy group is a homomorphic image of the fundamental group of the complement to the discriminant, all relations in $\mathcal{B}(\{\gamma_i\})$ must necessarily be preserved by $\rho_{\chi'}$, meaning the generators of $M_{\chi'}$ satisfy $\mathcal{B}(\{h_i\})$. The generators of the group $M_{\chi'}$ are complex reflections and as such have finite order. We denote the set of such relations $\mathcal{F}(\{h_i\}) = \{h_i^{\Gamma_i} = 1, i = 1, \ldots, \ell\}$, where Γ_i is the order of the only eigenvalue of h_i distinct from 1.

In a style similar to [7], we construct a short exact sequence:

$$\{1\} \to P_{\chi'} \to \pi_1(\Sigma') \overset{\rho_{\chi'}}{\to} M_{\chi'} \to \{1\},$$

where $P_{\chi'}$ is the kernel of the representation.

Definition 5.2. If non-generic points of $P \cap \Sigma$ are at most double points, the group $\pi_1(\Sigma')$ will be called a *braid group*, otherwise a *braid-like group*.

This corresponds with the definitions of braid relations and braid-like relations in Sections 2.2.2 to 2.2.4. Braid groups have generators satisfying braid relations, braid-like groups have generators satisfying braid relations and braid-like relations.

We observe in all our cases that the group $M_{\chi'}$ is an irreducible subgroup of $GL(k, \mathbb{C})$, so by Schur's lemma its centre is cyclic and generated by scalar matrices. We denote the centre of a group G by ZG. Consider the restriction of the short exact sequence of groups to their centres, also giving a short exact sequence [7]:

$$\{1\} \to ZP_{\chi'} \to Z\pi_1(\Sigma') \stackrel{\rho_{\chi'}}{\to} ZM_{\chi'} \to \{1\}.$$

Let C_σ denote an element $C_\sigma = \prod_{i=1}^k h_{\sigma(i)}$, where the order of C_σ is taken to be N (which has already been introduced) for a certain choice of σ. We choose one such C_σ and call it C.

Proposition 5.3. *The group $ZM_{\chi'}$ contains a power of C.*

Proof. The centre of a braid group is generated by all elements of the form $\zeta_\sigma = \left(\prod_{i=1}^k \gamma_{\sigma(i)}\right)^q$ where σ is a permutation of the set $\{1, \ldots, k\}$ providing an ordering of group generators γ_i, and q is the smallest positive integer such that the action of ζ on the strands of the braid(-like) group is equal to the identity.

Since $\rho_{\chi'}$ is a homomorphism, the element $\rho_{\chi'}(\zeta_\sigma) = \left(\prod_{i=1}^k h_{\sigma(i)}\right)^q = C_\sigma^q \in ZM_{\chi'}$. We choose one such element such that $\rho_{\chi'}(\zeta) = C^q$, for an element C.

□

We observe that this result does not depend on the choice of C, provided C is chosen to have order N.

Remark 5.4. Although Proposition 5.3 can be proved instantly by taking a trivial power of C, the proof given allows us to make the following conjecture.

Conjecture 5.5. *There are no elements in $ZM_{\chi'}$ other than those obtained by taking power of C.*

This implies that $\rho_{\chi'}^{-1}(ZM_{\chi'}) = Z\pi_1(\Sigma')$ and moreover $|ZM_{\chi'}|$ is finite. We label the generators so that $\sigma = id$ and $C = \prod_{i=1}^k h_i$. In each case the value of q so that $C^q \in ZM_{\chi'}$ can be found experimentally. We make the observation that q depends only on the skeleton of the Dynkin(-like) diagram, and values of q for specific skeletons are given in Table 5.1. In each case where the skeleton appears in the classification of discrete finite complex reflection groups, the values of q agree. For skeletons of diagrams that don't already appear in the literature, our values seem to generalise the previous results [7]. This is what allows us to make Conjecture 5.5.

In the invariant case, we construct a presentation consisting of all known relations so far. Presentations are ended with "..." to indicate the possibility of more relations. This possible incompleteness of our presentation means the following consideration does not depend on Conjecture 5.5.

$$M_{\chi'} = \left\langle h_1, \ldots, h_\ell | \mathcal{B}(h_i), \mathcal{F}(h_i), C^N = 1, \ldots \right\rangle.$$

The projectivised version has presentation

$$\mathbb{P}M_{\chi'} = \langle h_1, \ldots, h_\ell | \mathcal{B}(h_i), \mathcal{F}(h_i), C^q = 1, \ldots \rangle.$$

We must amend this definition slightly for the equivariant case. In the case that the only linear monomial preserved by the equivariant symmetry is x, we assume the basic equivariant has the form

$$g_x = \xi C^{\frac{b}{n}},$$

where ξ may be ι_I, the involution from Section 2.1.3, or the identity. Then we find experimentally that instead of N in our above definition, we should choose

$$N' = \frac{|g_x|}{|\xi|} = \frac{Nn}{\operatorname{ord}(\xi)b}.$$

This is since $g_x = \xi C^{w_f/(w_f - w_x)}$, where w_f, w_x are weights of f and x as defined in Example 2.1. This leaves three cases in which more than one linear term is preserved, namely $E_{12}/\mathbb{Z}_4$, $Q_{12}/\mathbb{Z}_6$ and $U_{12}/\mathbb{Z}_8$, which may be worked out directly using elementary matrix calculations with the group generators. The following table indicates what value we should take for N'

for each of the equivariant symmetric singularities, giving the presentation

$$M_{\chi'} = \left\langle h_1, \ldots, h_\ell | \mathcal{B}(h_i), \mathcal{F}(h_i), C^{N'} = 1, \ldots \right\rangle.$$

Singularity	N'
$E_{12}/\mathbb{Z}_9$	36
$E_{12}/\mathbb{Z}_{12}$	36
$E_{12}/\mathbb{Z}_4$	4
$Q_{12}/\mathbb{Z}_5$	15
$E_{14}/\mathbb{Z}_7$	21
$W_{12}/\mathbb{Z}_8$	16
$W_{12}/\mathbb{Z}_6$	15
$Q_{12}/\mathbb{Z}_9$	9
$Q_{12}/\mathbb{Z}_6$	12
$Q_{12}/\mathbb{Z}_4$	12
$U_{12}/\mathbb{Z}_8$	8

As stated previously, in the invariant case we simply choose $N' = N$. We further define the following groups, for which these relations stated so far are the only relations:

$$\begin{aligned} \overline{M}_{\chi'} &= \left\langle h_1, \ldots, h_\ell | \mathcal{B}(h_i), \mathcal{F}(h_i), C^{N'} = 1 \right\rangle, \\ \overline{\mathbb{P}M}_{\chi'} &= \langle h_1, \ldots, h_\ell | \mathcal{B}(h_i), \mathcal{F}(h_i), C^q = 1 \rangle. \end{aligned}$$

Conjecture 5.6. *For each group in our classification, there exists an isomorphism of groups $M_{\chi'} \cong \overline{M_{\chi'}}$, and therefore also of the groups $\mathbb{P}M_{\chi'} \cong \overline{\mathbb{P}M_{\chi'}}$.*

This is equivalent to saying the matrix group $M_{\chi'}$ is a faithful representation of the abstract group $\overline{M}_{\chi'}$.

Remark 5.7. The general philosophy is that a discrete group has few relations, a non-discrete group has many. Since we know the projectivised versions of our groups are discrete, we expect there to be few relations and this allows us to make our conjecture.

Remark 5.8. The projectivisations of all of the 2 dimensional groups do indeed satisfy this conjecture since they are discrete subgroups of $\mathbb{P}U(1,1)$ and are isomorphic to triangle groups acting on the Poincaré disk. These will be discussed more in Section 5.1.1. The 3 dimensional groups that can be identified within the literature in Section 5.1.2 also satisfy the conjecture.

We conclude this subsection with a further conjecture which is an immediate corollary of Conjecture 5.5 and Corollary 2.19.

Conjecture 5.9. *For each symmetry considered in this thesis, the resulting monodromy group $M_{\chi'}$ is a discrete subgroup of $U(k-1,1)$.*

Idea of Proof: We know from Corollary 2.19 that the group $\mathbb{P}M_{\chi'}$ is discrete. We find the projectivised group by taking the quotient by the centre

$$M_{\chi'} \xrightarrow{/ZM_{\chi'}} \mathbb{P}M_{\chi'}.$$

By Conjecture 5.5 $ZM_{\chi'}$ is finite, therefore the fibres of this map are finite.

5.1.1 2 Dimensional Groups

The group $M_{\chi'} \subset U(k-1,1) \subset GL(k,\mathbb{C})$ acts on the space $\mathbb{C}^k$ with appropriately chosen coordinates $z_0, \dots, z_{k-1}$ so that $M_{\chi'}$ preserves the Hermitian form $-|z_0|^2 + |z_1|^2 + \cdots + |z_{k-1}|^2$. In particular, $M_{\chi'}$ sends the cone

$$-|z_0|^2 + |z_1|^2 + \cdots + |z_{k-1}|^2 < 0$$

into itself. Setting $z_0 \neq 0$ defines an affine chart in $\mathbb{CP}^{k-1}$, in which the cone is given by

$$\left|\frac{z_1}{z_0}\right|^2 + \cdots + \left|\frac{z_{k-1}}{z_0}\right|^2 < 1.$$

The projectivisation $\mathbb{P}M_{\chi'}$ of $M_{\chi'}$ acts on $\mathbb{CP}^{k-1}$ preserving the projectivisation of the cone. In the affine chart $z_0 \neq 0$ and its coordinates $w_j = \frac{z_j}{z_0}$, this means that the action of $\mathbb{P}M_{\chi'}$ sends the hyperbolic ball

$$B = \left\{|w_1|^2 + \cdots + |w_k|^2 < 1\right\} \subset \mathbb{C}^{k-1}$$

Table 5.1: Central generators for general diagrams

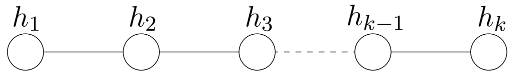
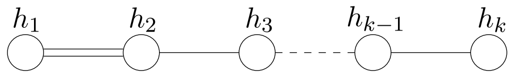
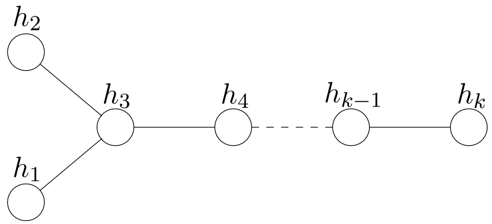
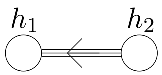
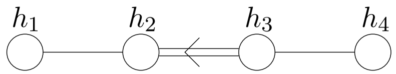
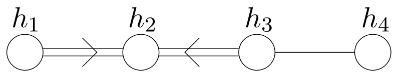
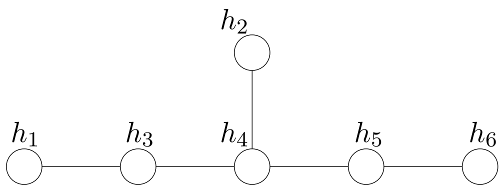
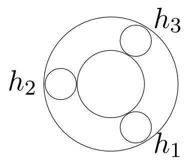
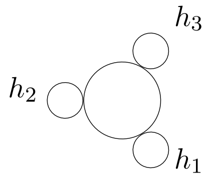
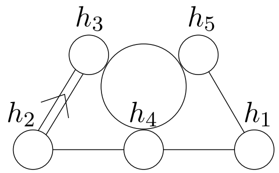
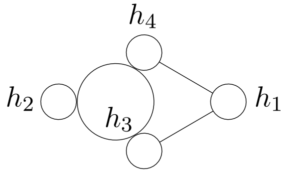
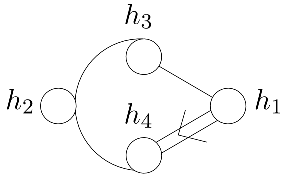
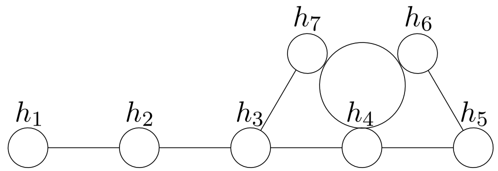
Diagram	q
	$k+1$
	k
	$\frac{2(k-1)}{(2\wedge k)}$
	3
	6
	5
	6
	12

Diagram	q
	3
	1
	5
	12
	3
	9

into itself.

For the rest of this section, we let $k = 2$ so that $B = \{|w_1|^2 < 1\} \subset \mathbb{C}$.

Definition 5.10. For a triple of positive integers $r_1 \leq r_2 \leq r_3$ such that $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} < 1$, there is a triangle in the Poincaré disk $\mathbb{H}$ with angles π/r_1, π/r_2, π/r_3, which is unique up to isometry. The hyperbolic reflections in the sides of the triangle generate a group $D(r_1, r_2, r_3)$ of isometries of $\mathbb{H}$, called a *hyperbolic triangle group*, which has the triangle Δ as fundamental domain.

Generating symmetries of this group are either holomorphic or anti-holomorphic.

Definition 5.11. We define the index 2 subgroup $D_+(r_1, r_2, r_3) \subset D(r_1, r_2, r_3)$ to be the subgroup consisting of holomorphic functions, and call it a *holomorphic triangle group.*

The fundamental domain Δ_+ is the union of two adjacent copies of Δ.

Theorem 5.12. *The projectivised groups $\mathbb{P}M_\chi$ coming from the two dimensional deformations in our classification are holomorphic triangle groups $D_+(r_1, r_2, r_3)$. Table 5.2 lists all such groups.*

The angles forming the hyperbolic triangles arise not only from the symmetries but also from the weights of the parameters in the versal deformation as defined on page 4. If the singularity has the alternative notation $X^{(m_1,m_2)}$, then the weights of basic invariants for the group X are equal to the weights of parameters according to Section 2.2.6.

The quotient B/Δ_+ is a sphere with 3 marked points corresponding to the $\mathbb{Z}_{r_i}$ stationary groups. This space is isomorphic to the weighted projectivisation of the base of versal deformation. The orders of the marked points come from the Picard-Lefschetz operators and the quasi-degrees of parameters in the g-versal deformation (cf. Section 2.2.6).

Example 5.13. Consider the singularity $E_{14}|\mathbb{Z}_8$. Discussed on Page 55, the Dynkin diagram is as follows.

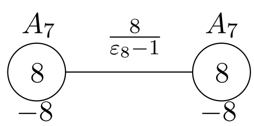


Table 5.2: Triangle Groups

Singularities	Alt. Notation	r_1, r_2, r_3
$E_{12}\|\mathbb{Z}_7$	$A_2^{(7)}$	$2, 3, 7$
$Q_{10}\|\mathbb{Z}_8, E_{14}\|\mathbb{Z}_8$	$A_2^{(8)}$	$2, 3, 8$
$E_{12}/\mathbb{Z}_9$	$B_2^{(3,9)}$	$2, 3, 9$
$Q_{12}\|\mathbb{Z}_{10}$	$A_2^{(10)}$	$2, 3, 10$
$Z_{11}\|\mathbb{Z}_{10}$	$G_2^{(10)}$	$2, 3, 10$
$U_{12}\|\mathbb{Z}_{12}$	$A_2^{(12)}$	$2, 3, 12$
$Q_{10}\|\mathbb{Z}_{12}, E_{14}\|\mathbb{Z}_{12}$	$B_2^{(3,12)}$	$2, 3, 12$
$Z_{13}\|\mathbb{Z}_9$	$B_2^{(3,18)}$	$2, 3, 18$
$S_{11}\|\mathbb{Z}_8, W_{13}\|\mathbb{Z}_8, W_{12}/\mathbb{Z}_8$	$B_2^{(4,8)}$	$2, 4, 8$
$W_{12}\|\mathbb{Z}_{10}$	$B_2^{(5)}$	$2, 5, 5$
$U_{12}\|\mathbb{Z}_6$	$B_2^{(6,6)}$	$2, 6, 6$
$E_{17}/\mathbb{Z}_7$	$G_2^{(7,14)}$	$2, 7, 14$
$U_{12}\|\mathbb{Z}_{12}$	$G_2^{(4)}$	$3, 4, 4$

The intersection matrix of this singularity is

$$\begin{pmatrix} -8 & \frac{8}{\varepsilon_8 - 1} \\ \frac{8}{\overline{\varepsilon_8} - 1} & -8 \end{pmatrix},$$

and the Picard-Lefschetz operators are

$$h_1 = \begin{pmatrix} \varepsilon_8 & \varepsilon_8 \\ 0 & 1 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 1 & 0 \\ -1 & \varepsilon_8 \end{pmatrix}.$$

The projectivised version of this group acts on $\mathbb{CP}^1$ with coordinate $Z = (z : 1)$. The interior of the Poincaré disk $\mathbb{H}$ representing the points in the chart is given by

$$\begin{pmatrix} z & 1 \end{pmatrix} \begin{pmatrix} -8 & \frac{8}{\varepsilon_8 - 1} \\ \frac{8}{\overline{\varepsilon_8} - 1} & -8 \end{pmatrix} \begin{pmatrix} \overline{z} \\ 1 \end{pmatrix} > 0.$$

This simplifies to

$$\left|z - \frac{1}{\overline{\varepsilon_8} - 1}\right|^2 < \left|\frac{1}{\overline{\varepsilon_8} - 1}\right|^2 - 1 = \frac{\sqrt{2}}{2}.$$

Let p_1 be the projective point in this chart fixed by the generator h_1. That is, p_1 is a solution of $h_1 Z = Z$. We find the only solution in our chart is

$$p_1 = \frac{1}{\overline{\varepsilon_8} - 1}.$$

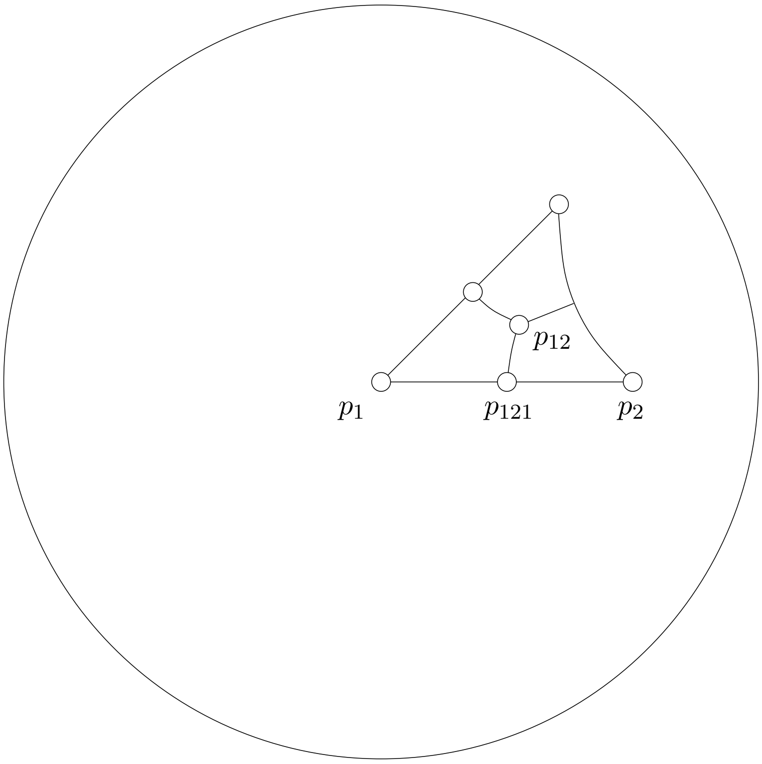
This is at the centre of our chart. The other solution is $z = \infty$. When we find roots of $h_2 Z = Z$, we find two solutions: $z = 0$, and $z = \varepsilon_8 - 1$. The latter of these is inside the disk corresponding to this chart, so we choose

$$p_2 = \varepsilon_8 - 1.$$

We let p_{12}, p_{121} denote fixed points of elements $h_1 h_2$, $h_1 h_2 h_1$ respectively in this chart. We find

$$\begin{aligned} p_{12} &= \frac{\varepsilon_8(1 + i\sqrt{3})}{2}, \\ p_{121} &= \varepsilon_{16}^5. \end{aligned}$$

The points mentioned are noted on the picture below, the fundamental domain Δ_+ is any of the 3 kites shown.



5.1.2 3 Dimensional Groups

In [18] there is a survey of discrete, complex projective subgroups of $\mathbb{P}U(2,1)$. In particular it contains the Mostow groups

$$\Gamma_{p,k} = \left\langle J, A, R : \begin{array}{c} J^3 = A^k = R^p = 1, \\ A = (JR^{-1}J)^2, AR = RA \end{array} \right\rangle,$$

where permitted values for p, k are as follows.

p	3	3	3	4	4	5	5	6	6
k	4	5	6	3	4	2	3	2	3

Consider the family of groups arising in our classification:

$$\mathbb{P}B_3^{(k,p)} = \left\langle h_1, h_2, h_3 : \begin{array}{c} h_1^k = h_2^p = h_3^p = (h_1h_2h_3)^3 = 1 \\ (h_1h_2)^2 = (h_2h_1)^2, h_2h_3h_2 = h_3h_2h_3, h_1h_3 = h_3h_1 \end{array} \right\rangle,$$

where p, k are the following.

p	3	3	5	6
k	4	6	2	2

Proposition 5.14. *For such groups appearing in this classification, there exists an isomorphism*

$$\Gamma_{p,k} \cong \mathbb{P}B_3^{(k,p)}.$$

Proof. An isomorphism is

$$\begin{aligned} J &= h_3 h_2 h_1 \\ A &= h_1 \\ R &= h_3. \end{aligned}$$

□

This covers the groups

$$\begin{aligned} Q_{11}|\mathbb{Z}_6 &\cong C_3^{(2,6)} \\ Z_{13}|\mathbb{Z}_6 &\cong B_3^{(6,3)} \\ E_{13}|\mathbb{Z}_{10} &\cong C_3^{(2,5)} \\ E_{12}/\mathbb{Z}_{12} &\cong B_3^{(4,3)}. \end{aligned}$$

We have so far been unable to identify other projectivised 3 dimensional groups within the literature. In [18], there are two more families of projectivised three dimensional groups; one of which is well presented, and the other has four generators. It is possible that these groups correspond to groups appearing in this thesis, but no isomorphism is known.

Chapter 6

Appendix

6.1 Diophantine equation solver

```
###########################################
# Program to solve Diophantine Equations #
# with four unknown variables to find    #
###########################################

# Declare variables
a:='a':b:='b':c:='c':d:='d':
# Define conditions to check equal zero
cond1:=<declare function>;
cond2:=<declare function>;
# Set iteration bounds per variable (must be positive)
MAXP:=<declare constant>;
# Loop around all combinations of all values of each variable
# within range
a:=-MAXP: while a<=MAXP do
 b:=-MAXP: while b<=MAXP do
  c:=-MAXP: while c<=MAXP do
   d:=-MAXP: while d<=MAXP do
# Output values of variables if conditions are satisfied
```

```
    if(cond1=0 and cond2=0) then print([a,b,c,d,cond1,cond2]);
    end if;
    d:=d+1:
   od:
   c:=c+1:
  od:
  b:=b+1:
 od:
 a:=a+1:
od:
# End of program
```

6.2 Approximate Diophantine equation solver

```
##############################################################
# Program to approximate solutions to Diophantine Equations #
# with four unknown variables to find                       #
##############################################################

# Declare variables
a:='a':b:='b':c:='c':d:='d':
#Define polynomial
poly:=<declare polynomial>;
# Define constant polynomial should equal
value:=<declare constant>;
# Set iteration bounds per variable (must be positive)
MAXP:=<declare constant>;
# Greatest permitted error negative index
E:=<declare constant>;
# Loop around all combinations of all values of each variable
# within range
a:=-MAXP: while a<=MAXP do
 b:=-MAXP: while b<=MAXP do
```

```
  c:=-MAXP: while c<=MAXP do
   d:=-MAXP: while d<=MAXP do
# Output values of variables if conditions are satisfied
    TEST:=evalf(abs(poly-value));
    if(TEST<10^(-E)) then print([a,b,c,d,TEST]); end if;
    d:=d+1:
   od:
   c:=c+1:
  od:
  b:=b+1:
 od:
 a:=a+1:
od:
```

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