

# Infinite families of monohedral disk tilings

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## 1 Introduction

A tiling of a planar shape is called *monohedral* if all tiles are congruent to each other. We will investigate the possibility of producing monohedral tilings of the disk. Such tilings are produced on a daily basis by pizza chefs by taking radial cuts distributed evenly around the centre of the pizza. After constructing this tiling, a neighbourhood of the origin has non-trivial intersection with each tile. This brings us to the main question of this article:

*Can we construct monohedral tilings of the disk such that a neighbourhood of the origin has trivial intersection with at least one tile?*

The answer to this question is **yes!** The particular solution given in Figure 2 appears as the logo of the Mass Program at Penn State [1]. On their website it is stated:

*We do not know whether there is a partition of the disc into congruent pieces other than the logo such that not all pieces contain the center and not all boundaries are circle arcs.*

This also hints at the knowledge of a further family of solutions which are displayed in Figure 5. This article presents a new family of solutions to this problem which generalise the solution in Figure 2. There is no claim that these two families contain all solutions to this problem.

This introduction has purposely been lacking in images so that the reader has ample opportunity to investigate this fun problem before continuing!

A specific case of the problem should be mentioned: the problem of producing a monohedral disk tiling such that a neighbourhood of the origin is contained entirely within the interior of a single tile. This was presented in [2] as an unsolved problem, and may be impossible (e.g. [3]).

## 2 Previously known solutions

A known solution to this problem may be found by slightly modifying a very elementary tessellation of regular polygons. Figure 1 shows how we may construct a regular hexagon by placing 6 equilateral triangles around a point which becomes the centre of the regular hexagon. An equilateral triangle has three lines of symmetry. We may include these lines in Figure 1 in such a way that cyclic symmetry is preserved, but reflective symmetry is not.

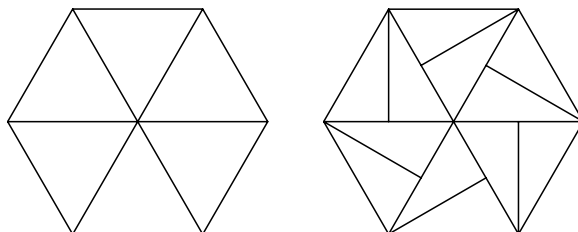


Figure 1: A regular hexagon tiled by 6 equilateral triangles, and 12 congruent scalene triangles.

It follows that this tiling is a solution to a similar problem thinking of the hexagon rather than the disk, since 6 out of the 12 tiles do not touch the centre of the hexagon.

We return to tiling the disk with congruent tiles. Since some tiles touch the boundary of the disk, all tiles must have at least one edge which has the same curvature. Figure 2 show a modification of the hexagonal tiling in Figure 1, created by curving all edges in a cyclic way so that each edge has the same curvature as the perimeter of the disk. We refer to tiles of this type as *shields*. In an absolutely similar manner to that of the hexagon, we now proceed to split each shield in half down its line of symmetry. Note that we are forced to do this in a cyclic way according to the curvature of the interior edges. Since 6 out of the 12 half-shields don't touch the centre of the disk, Figure 2 gives a solution to the tiling problem. We call this tiling a *tiling of type 1*, denoted  $T_{12}^1$ , where  $T_j^i$  denotes a tiling of type  $i$  containing  $j$  tiles.

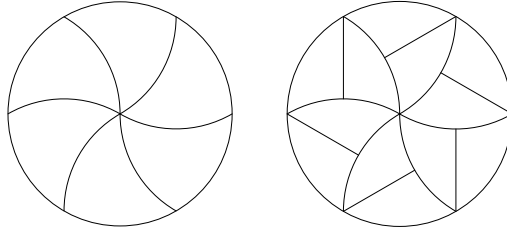


Figure 2: A disk tiled by 6 shields, and 12 half-shields ( $T_{12}^1$ ).

As well as being a curved version of the tessellation of the hexagon by equilateral triangles, this is also a curved version of part of a non-periodic tiling of irregular quadrilaterals found in [4]. The part of this infinite planar tiling we are interested in is shown in Figure 3. This object also appears on page 236 of [5] in the setting of piano-hinged dissections of regular dodecagons.

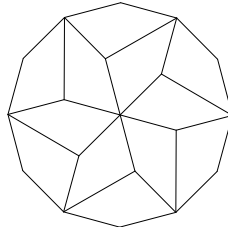


Figure 3: Part of a non-periodic planar tiling by an irregular quadrilateral.

We continue the classification. Figure 4 shows how the tiling of 6 shields may be considered to be a sub-tiling of a cyclic tiling of order 12 with all tiles congruent. Highlighted in this figure is the shield. The shield has a line of symmetry, so we can flip each pair of tiles within each of the 6 shields getting a second solution to the problem. We call this a *tiling of type 2*, denoted by  $T_{12}^2$ .

The tiling  $T_{12}^2$  was constructed by splitting each shield into 2 parts in a natural way. If we similarly split each shield into  $n$  parts ( $n \geq 2$ ), we get a tiling which we denote  $T_{6n}^2$  after flipping each  $n$ -tuple of tiles within each of the 6 shields. Figure 5 displays some of these tilings.

In fact, each  $T_{6n}^2$  represents a set of tilings. Some members of  $T_{12}^2$  are shown in Figure 6. The first of these is similar to a tiling appearing in the discussion [6] as a counter example to the conjecture that all tilings solving this problem should have non-trivial cyclic symmetry.

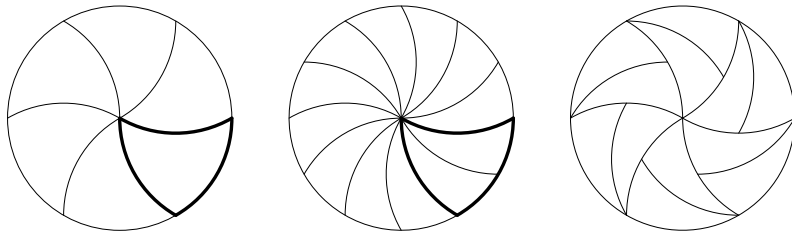


Figure 4: A disk tiled by 6 shields, and two tilings with 12 tiles ( $T_{12}^2$ ).

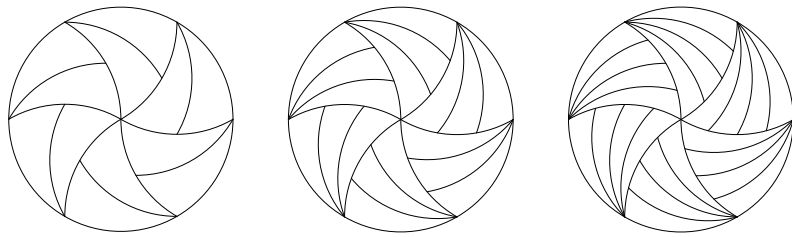


Figure 5: Tilings  $T_{12}^2$ ,  $T_{18}^2$ ,  $T_{24}^2$ .

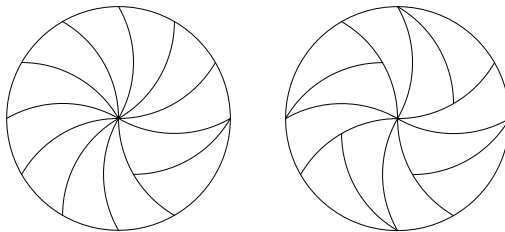


Figure 6: Tilings belonging to the set  $T_{12}^2$ .

The tilings  $T_{12}^1$  and  $T_{12}^2$  are widely known. The tiling  $T_{6n}^2$  is not obviously present in the literature but is a generalisation of  $T_{12}^2$ .

In section 3 we present a construction of a new family of solutions generalising  $T_{12}^1$ .

### 3 A new family of solutions

In this section we show how the tiling  $T_{12}^1$  may be generalised to a tiling  $T_{4n}^1$ , where  $n \geq 3$  is odd.

Let  $n \geq 5$  be an odd number and take a regular  $n$ -gon. Choose  $\frac{n+1}{2}$  connected edges, and split these from the rest of the edges. The remaining  $\frac{n-1}{2}$  edges may then be reattached to produce a concave polygon. We call this shape a *regular reflex polygon*. For example, see Figure 7. We define a regular reflex 3-gon to be an equilateral triangle.

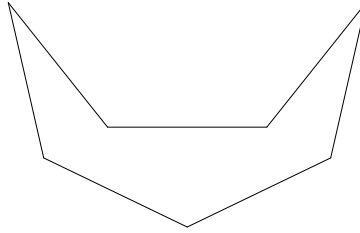


Figure 7: A regular reflex 7-gon.

If we arrange two regular reflex  $n$ -gons such that the acute angles touch, this picture is indistinguishable from an intersection of two regular  $n$ -gons. This demonstrates that a regular reflex  $n$ -gon has a line of symmetry (Figure 8).

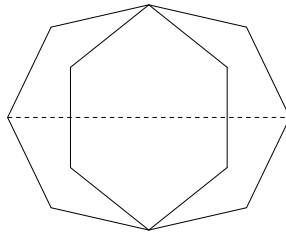


Figure 8: A regular reflex  $n$ -gon has a line of symmetry.

Regular reflex  $n$ -gons tessellate naturally. Indeed, a tiling of regular reflex 7-gons can be seen in Figure 9. The original source of this picture is page 515 of [7].

The  $\frac{n-1}{2}$  obtuse angles of a regular reflex  $n$ -gon are each  $\frac{n-2}{2n}$  of a full turn, since they are the interior angles of a regular  $n$ -gon. The  $\frac{n-3}{2}$  reflex angles of a regular reflex  $n$ -gon are each  $\frac{n+2}{2n}$  of a full turn since they are the exterior angles of a regular  $n$ -gon. The sum of the interior angles for any regular  $n$ -gon is equal to  $\frac{n-2}{2}$ . Let  $\alpha$  denote the size of each acute angle of the regular reflex  $n$ -gon. Then  $\alpha$  satisfies the equation

$$\frac{n-1}{2} \frac{n-2}{2n} + \frac{n-3}{2} \frac{n+2}{2n} + 2\alpha = \frac{n-2}{2}.$$

This simplifies to a linear equation with the unique solution  $\alpha = \frac{1}{2n}$  of a full turn.

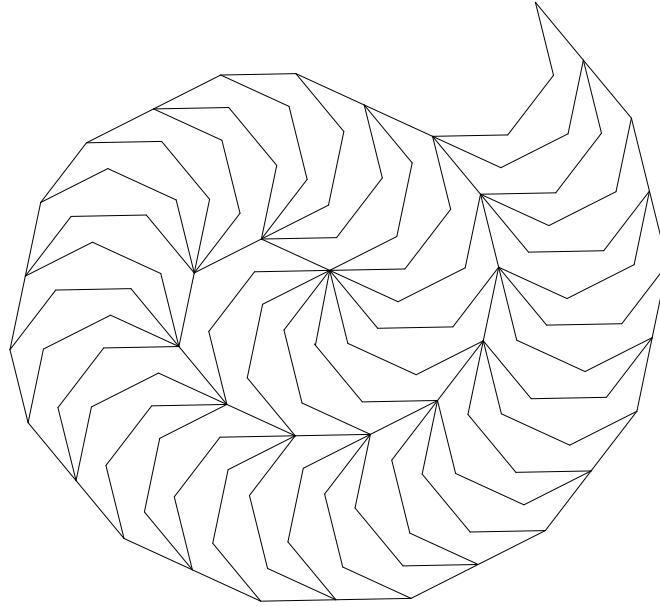


Figure 9: A tessellation of reflex 7-gons.

We conclude that  $2n$  regular reflex  $n$ -gons may be tessellated cyclically around a given vertex adjacent to one of the acute angles, producing a tiling of a regular  $2n$ -gon. Since a regular reflex  $n$ -gon has a line of symmetry, each tile may be split in two to produce a monohedral tiling of the regular  $2n$ -gon with  $4n$  tiles. This is shown in Figure 10 for  $n = 5$ .

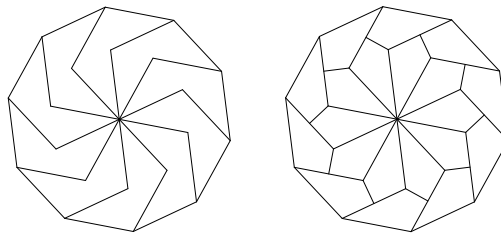


Figure 10: A regular decagon tiled by regular reflex pentagons and half regular reflex pentagons.

Consider now a tiling consisting of 3 reflex  $n$ -gons, in which three of the acute angles meet at a vertex  $C$ . This is shown in Figure 11 for  $n = 7$ . Let  $e_1, e_2, e_3$  be edges of the respective regular  $n$ -gons forming a connected subset of the boundary of this tiling.

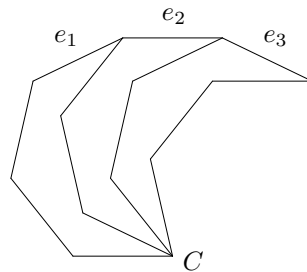


Figure 11: Three regular reflex 7-gons.

Replace  $e_1, e_2, e_3$  by arcs  $a_1, a_2, a_3$ , each with centre of curvature at  $C$ . For the resulting tiles to be symmetric, the opposite edges of each tile must be replaced with arcs with the same curvature. But the resulting tiles are not congruent, and more edges should be replaced with similar arcs to achieve this. Continue replacing edges with arcs in this fashion until all tiles are congruent and each has a line of symmetry. This process is demonstrated in Figure 12, again for  $n = 7$ , where the dotted lines represent the desired lines of symmetry of tiles in the first two diagrams, and genuine lines of symmetry of tiles in the third.

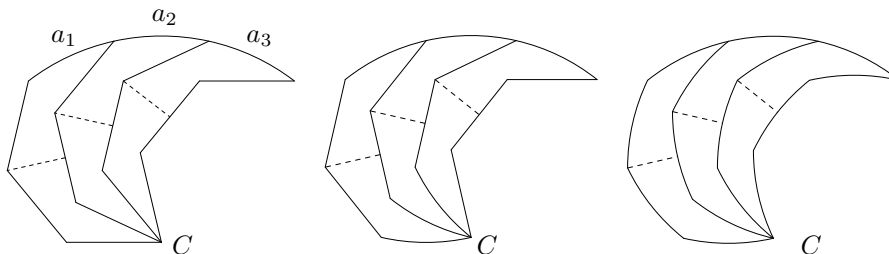


Figure 12: Replacing edges with arcs.

Since the acute angle of a regular reflex  $n$ -gon is  $\frac{1}{2n}$  of a full turn, we continue tiling about the vertex  $C$  with congruent tiles to produce a tiling with  $2n$  tiles. Since the edges on the boundary of the tiling have  $C$  as a common centre of curvature, the boundary must be a circle with centre  $C$ , and we have produced a monohedral tiling of the disk. Each of the  $2n$  tiles has a line of symmetry. A monohedral tiling with  $4n$  tiles is obtained by cutting along these lines of symmetry. Half of the resulting tiles do not touch the centre of the disk. Some members of this family are shown in Figure 13.

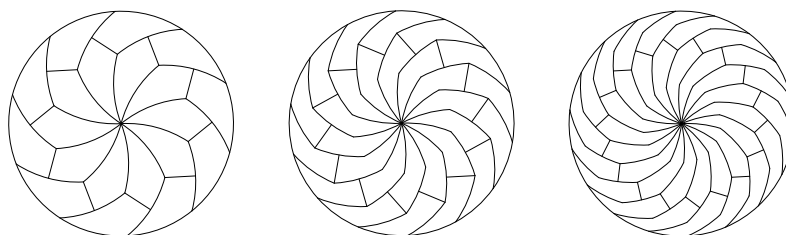


Figure 13: Tilings  $T_{20}^1, T_{28}^1, T_{36}^1$ .

We consider these tilings as generalisations of  $T_{12}^1$  since replacing the edges of a regular reflex 3-gon (an equilateral triangle) with arcs in a similar manner produces the shield which was shown in Figure 2 in Section 2.

## 4 Acknowledgements

The author would like to acknowledge Colin Wright's observation that the new solutions in Figure 13 generalise the original solution in Figure 2, and thank him and Ian Porteous for many enthusiastic discussions about this problem (read Colin's Blog post about this topic at [8]). Thanks also to Karene Chu for the initial introduction to this problem.

## References

- [1] Mass program at Penn State: <http://www.math.psu.edu/mass/logo/>

- [2] H. T. Croft, K. J. Falconer, R. K. Guy, *Unsolved Problems In Geometry*, New York, Springer-Verlag (1991).
- [3] Math Overflow, *Is it possible to dissect a disk into congruent pieces, so that a neighborhood of the origin is contained within a single piece?*: <http://bit.ly/dSilb4>
- [4] M. Goldberg, *A dissection problem for sets of polygons*, Amer. Math. Monthly **71** (1964), 1077-1095.
- [5] G. N. Frederickson, *Piano-hinged dissections: time to fold!*, A K Peters/CRC Press (2006).
- [6] XKCD.com forums, *Unusual Cake Slicing*: <http://forums.xkcd.com/viewtopic.php?f=3&t=27963>
- [7] B. Grunbaum, G. C. Shephard, *Tilings And Patterns*, W. H. Freeman & Company (1986).
- [8] Solipsys, Colin Wright's Blog: <http://www.solipsys.co.uk/new/DissectingACircle.html?ColinsBlog>

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