Invariant Symmetries of Unimodal Function Singularities

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To the memory of Vladimir Igorevich Arnold

Abstract

We classify finite order symmetries g of the 14 exceptional unimodal function singularities f in 3 variables, which satisfy a so-called splitting condition. This means that the rank 2 positive subspace in the vanishing homology of f should not be contained in one eigenspace of $g_\star$. We also obtain a description of the hyperbolic complex reflection groups appearing as equivariant monodromy groups acting on the hyperbolic eigensubspaces arising.

One of the most famous classical results in singularity theory is the Arnold and Brieskorn discovery of the close relationship between simple function singularities and Weyl groups A_μ, D_μ, E_μ [1, 6]. A few years after it, Arnold extended the relationship to simple singularities with the $\mathbb{Z}_2$ reflection symmetry and Weyl groups B_μ, C_μ, F_4 [2] (see also Slodowy's book [24]).

Consideration of $\mathbb{Z}_m$ symmetries of simple functions led in [10, 11, 12, 25] to the appearance of Shephard-Todd groups within function singularities. The emphasis there was on realisations of the complex reflection groups as equivariant monodromy groups acting on the appropriate character subspaces in the homology of invariant Milnor fibres, and on the diffeomorphisms between the discriminants of the reflection groups and of the $\mathbb{Z}_m$-equivariant functions.

A further series of papers [13, 14, 15], on cyclic symmetries of the parabolic functions, brought in similar singularity realisations of certain complex crystallographic groups [22].

In this paper, we are naturally expanding the programme to cyclic symmetries of the 14 exceptional unimodal function singularities on one hand, and complex hyperbolic reflection groups on the other. The basic idea is as follows. In the 3-variable case, the intersection form on the vanishing homology of an exceptional unimodal function f is non-degenerate and has positive signature 2. Assume g is an automorphism of $\mathbb{C}^3$ of finite order m, and our function is g-invariant. Then g acts on the second homology of the Milnor fibre $f^{-1}(\varepsilon)$, and decomposes it into a direct sum of the character subspaces H_χ, $\chi^m = 1$, on which g acts as multiplication by χ. Assume the rank 2 positive subspace of the intersection form splits between two character summands. Then the monodromy within a g-invariant versal deformation of f acts as a complex hyperbolic reflection group on each of them. Developing further the technique introduced in papers on cyclically symmetric functions [10, 11, 12], we construct vanishing bases in the hyperbolic summands and obtain the generating reflections as the corresponding Picard-Lefschetz operators.

The main result of the paper is a complete classification of the invariant symmetries of the 14 singularities, which split the positive subspace in the vanishing homology, and the description – via constructing the corresponding Dynkin diagrams – of the complex hyperbolic groups arising. All the rank 2 reflection groups obtained projectivise to the triangle groups of the Poincaré disk. The task of identification of higher dimensional groups is left for a future paper, along with the consideration of the equivariant symmetry setting. It should be noted that it is the first time when complex hyperbolic reflection groups are appearing in a singularity theory context. The approach introduced may be useful for constructing new complex hyperbolic lattices (cf. [8, 20]).

The paper is organised as follows. Section 1 introduces the notion of singularities with symmetry, recalls the definitions and constructions given in [10, 11, 12]. Section 2 contains classification of splitting invariant symmetries of the 14 singularities. In Section 3.3 we construct Dynkin diagrams of the hyperbolic monodromy groups associated with the symmetric functions. Projectivisations of the rank 2 monodromy groups are considered in Section 4. More details of the constructions may be found in [16].

1 Singularities with symmetry

1.1 Symmetries and deformations

Our main objects of study will be pairs (f, g) consisting of a holomorphic function germ $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ with an isolated singularity, and a finite order automorphism g of $(\mathbb{C}^{n+1}, 0)$ under which f is invariant: $f \circ g = f$. The automorphism g will be called a *symmetry* of the function.

Assume the coordinates $x_0, \ldots, x_n$ in $(\mathbb{C}^{n+1}, 0)$ are chosen so that g is a diagonal linear transformation. Consider a deformation

$$f + \sum_{i=1}^{k} \lambda_i \varphi_i \tag{1}$$

of the function f, where the λ_i are parameters, and $\{\varphi_1, \ldots, \varphi_k\}$ is the set of all g-invariant elements of a monomial basis of the local ring Q_f of f. In the standard sense, deformation (1) is a g-miniversal deformation of f (see, for example [28]).

All through the paper, we use notation ε_m for $e^{2\pi i/m}$, and reserve ω for ε_3.

Example 1.1. Let f be a quasihomogeneous function of degree N with respect to the positive integer weights $w_0, \ldots, w_n$ of the coordinates x_j on $\mathbb{C}^{n+1}$. Assume $gcd(w_0, \ldots, w_n) = 1$, and consider the transformation

$$C: \quad x_j \mapsto \varepsilon_N^{w_j} x_j, \qquad j = 0, \ldots, n,$$

of $\mathbb{C}^{n+1}$. This corresponds to the values of f making one full anti-clockwise rotation in $\mathbb{C}$ about the origin. The transformation C is an order N symmetry of f. Take for an invariant symmetry g of f a power of C that has order m: $g = C^p$, $g^m = id$. Then the φ_i in (1) are exactly those elements of a monomial basis of Q_f whose degrees are divisible by m (cf. [25, 26]).

We shall use the notation Λ for the base of a g-miniversal deformation of a function f.

Definition 1.2. The *discriminant* $\Sigma \subset \Lambda$ of f is the set of all values $\lambda \in \Lambda$ of the parameters for which the members of its g-versal family have critical value 0.

Since a non-zero constant function is g-invariant, the discriminant is a hypersurface in Λ.

In what follows we will be working with representatives of germs of functions and sets we have introduced, but we will be still denoting them by the same letters.

1.2 Symmetric Milnor fibre and its equivariant monodromy

We define a Milnor fibre of a g-invariant function f following the usual approach (see [4, 5, 10]), as the intersection of a sufficiently small ball in $\mathbb{C}^{n+1}$ centred at the origin with the zero level of a generic member of an appropriate representative of a g-versal family F of f.

Let us fix a generic point $\star \in \Lambda \setminus \Sigma$. The Milnor fibre $V_\star$ is homotopic to a wedge of μ n-spheres [19], where μ is the Milnor number of f. A symmetry g sends $V_\star$ into itself. Therefore, its nth homology, of total rank μ, is a direct sum of *character subspaces*

$$H_n(V_\star, \mathbb{C}) = \oplus_{\chi^m = 1} H_\chi, \tag{2}$$

where m is the order of the automorphism g, and g acts as multiplication by χ on H_χ.

There is a standard way to define elements of the H_χ analogous to the ordinary Morse vanishing cycles. Namely, let W be the quotient of the fibre $V_\star$ by the action of the group $\mathbb{Z}_m$ generated by g, and $W' \subset W$ its subset of irregular orbits. Since all functions F_λ in the family F are g-invariant, a path in $\Lambda \setminus \Sigma$ from the point $\star$ to a generic point of the discriminant defines – at least in all our cases – a vanishing cycle $\sigma \in H_n(W, W'; \mathbb{Z})$, that is, a relative cycle which contracts to a point on the approach to the discriminant (cf. [2, 10, 11, 12]). The inverse image of this relative cycle in $V_\star$ consists of m cells $\sigma_0 \ldots, \sigma_{m-1}$, with the orientation inherited from σ, and ordered in the cyclic way:

$$g(\sigma_i) = \sigma_{(i+1) \bmod m}.$$

For appropriate values of χ, and in all the cases which will follow, the linear combination

$$\sigma_\chi = \sum_{i=0}^{m-1} \overline{\chi}^i \sigma_i$$

is a cycle, and thus provides an element of H_χ. We call σ_χ a *vanishing χ-cycle*.

The monodromy representation of the fundamental group $\pi_1(\Lambda \setminus \Sigma, \star)$ on $H_n(V_\star, \mathbb{C})$ is a direct sum of the representations on the individual summands H_χ. We denote the corresponding monodromy groups M_χ.

Depending on the parity of n, the intersection form on $H_n(V_\star, \mathbb{Z})$ naturally extends to $H_n(V_\star, \mathbb{C})$ in either an Hermitian or skew-Hermitian way. Assume that a vanishing χ-cycle σ_χ has a non-zero self-intersection number $\langle \sigma_\chi, \sigma_\chi \rangle$. Then according to [4, 7, 10] the related Picard-Lefschetz operator in M_χ is

$$h_\chi : \quad c \mapsto c + (e-1)\frac{\langle c, \sigma_\chi \rangle}{\langle \sigma_\chi, \sigma_\chi \rangle}\sigma_\chi,$$

where e is the eigenvalue of the operator on σ_χ. This is a (skew-)Hermitian reflection on H_χ.

To obtain a generating set of an M_χ, we proceed in the traditional manner. For this, we start with a generic line $L \subset \Lambda$ passing through the base point $\star$. Let $c_1, \ldots, c_r$ be the points at which L meets Σ. We choose a *distinguished system of paths* on L, that is, paths $\gamma_1, \ldots, \gamma_r$ in L, starting at $\star$ and leading to the c_i, which have no self- and mutual intersections except for the point $\star$ itself. The Picard-Lefschetz operators $h_{i,\chi}$ on the H_χ corresponding to the paths of the system generate the M_χ. Thus knowledge of the eigenvalues of the $h_{i,\chi}$ and of the intersection numbers of the χ-cycles vanishing at $c_1, \ldots, c_r$ yields a description of the monodromy group M_χ.

2 Exceptional unimodal functions

2.1 The list of singularities

Now assume that f is one of the 14 exceptional unimodal singularities (see, for example, [3, 5]) and $n = 2$. Table 1 gives a normal form of the quasihomogeneous member of each of the 14 one-parameter families, along with the weights of the coordinates. The weights are chosen so that $gcd(w_x, w_y, w_z) = 1$. For such choice, the degree N of each of the 14 quasihomogeneous singularities coincides with the order of its classical monodromy, which is usually called the *Coxeter number* of the singularity. Therefore, we will refer to N as the Coxeter number of the function. Respectively, the transformation

$$C : (x, y, z) \mapsto (\varepsilon_N^{w_x} x, \varepsilon_N^{w_y} y, \varepsilon_N^{w_z} z)$$

from Example 1.1 will be called the *Coxeter transformation* of the function.

Table 1 also gives one of possible choices of a monomial basis of the local ring of each of the singularities, and the weights of its elements.

The subscript in the notation of a singularity is its Milnor number μ. Pairs of functions with the same Coxeter number are dual in the sense of Arnold. Any function with $\mu = 12$ is self-dual.

An arbitrary member of a unimodal family is obtained by addition to the table normal form of a multiple of its Hessian, that is, of a multiple of the versal monomial of top weight.

Assume we have two coordinate spaces, $\mathbb{C}^p_{u_1,\ldots,u_p}$ and $\mathbb{C}^q_{v_1,\ldots,v_q}$, with coordinates of positive integer weights $a_1, \ldots, a_p$ and $b_1, \ldots, b_q$. Then the space of map-germs from $(\mathbb{C}^p, 0)$ to $(\mathbb{C}^q, 0)$ has a natural grading: a monomial summand $u_1^{\alpha_1} \ldots u_p^{\alpha_p}$ in the jth coordinate function is assigned grading $\alpha_1 a_1 + \cdots + \alpha_p a_p - b_j$. For example, a quasihomogeneous automorphism g of $\mathbb{C}^p$ has all its monomial terms of grading 0. The determinant $Jac(g)$ of the Jacobi matrix of such automorphism is a non-zero constant, which is easily seen if the coordinates are ordered by the increase of their weights.

In what follows, we are restricting our attention to quasihomogeneous symmetries of exceptional unimodal singularities.

Table 1: Exceptional unimodal singularities

type and normal form	w_x	w_y	w_z	N	versal monomials and their weights													
E_{12}	14	6	21	42	1	y	y^2	x	y^3	xy	y^4	xy^2	y^5	xy^3	xy^4	xy^5	
$x^3+y^7+z^2$					0	6	12	14	18	20	24	26	30	32	38	44	
Z_{11}	8	6	15	30	1	y	x	y^2	xy	x^2	y^3	xy^2	y^4	xy^3	xy^4		
$x^3y+y^5+z^2$					0	6	8	12	14	16	18	20	24	26	32		
E_{13}	10	4	15	30	1	y	y^2	x	y^3	xy	y^4	xy^2	y^5	xy^3	y^6	y^7	y^8
$x^3+xy^5+z^2$					0	4	8	10	12	14	16	18	20	22	24	28	32
Q_{10}	9	8	6	24	1	z	y	x	z^2	yz	xy	z^3	yz^2	yz^3			
$x^2z+y^3+z^4$					0	6	8	9	12	14	17	18	20	26			
E_{14}	8	3	12	24	1	y	y^2	x	y^3	xy	y^4	xy^2	y^5	xy^3	y^6	xy^4	xy^5	xy^6
$x^3+y^8+z^2$					0	3	6	8	9	11	12	14	15	17	18	20	23	26
Z_{12}	6	4	11	22	1	y	x	y^2	xy	x^2	y^3	xy^2	y^4	xy^3	y^5	y^6	
$x^3y+xy^4+z^2$					0	4	6	8	10	12	12	14	16	18	20	24	
W_{12}	5	4	10	20	1	y	x	y^2	xy	x^2	y^3	xy^2	x^2y	xy^3	x^2y^2	x^2y^3	
$x^4+y^5+z^2$					0	4	5	8	9	10	12	13	14	17	18	22	
Q_{11}	7	6	4	18	1	z	y	x	z^2	yz	z^3	xy	yz^2	z^4	z^5		
$x^2z+y^3+yz^3$					0	4	6	7	8	10	12	13	14	16	20		
Z_{13}	5	3	9	18	1	y	x	y^2	xy	y^3	x^2	xy^2	y^4	xy^3	y^5	xy^4	xy^5
$x^3y+y^6+z^2$					0	3	5	6	8	9	10	11	12	14	15	17	20
S_{11}	5	4	6	16	1	y	x	z	y^2	xy	yz	y^3	xy^2	y^2z	y^3z		
$x^2z+yz^2+y^4$					0	4	5	6	8	9	10	12	13	14	18		
W_{13}	4	3	8	16	1	y	x	y^2	xy	x^2	y^3	xy^2	x^2y	y^4	x^2y^2	y^5	y^6
$x^4+xy^4+z^2$					0	3	4	6	7	8	9	10	11	12	14	15	18
Q_{12}	6	5	3	15	1	z	y	z^2	x	yz	z^3	yz^2	xy	z^4	yz^3	yz^4	
$x^2z+y^3+z^5$					0	3	5	6	6	8	9	11	11	12	14	17	
S_{12}	4	3	5	13	1	y	x	z	y^2	xy	yz	y^3	z^2	y^2z	y^4	z^3	
$x^2z+yz^2+xy^3$					0	3	4	5	6	7	8	9	10	11	12	15	
U_{12}	4	4	3	12	1	z	x	y	z^2	xz	yz	xy	xz^2	yz^2	xyz	xyz^2	
$x^3+y^3+z^4$					0	3	4	4	6	7	7	8	10	10	11	14	
U_{12}	4	4	3		1	z	x	y	z^2	xz	yz	y^2	xz^2	yz^2	y^2z	y^2z^2	
$x^2y+y^3+z^4$					0	3	4	4	6	7	7	8	10	10	11	14	

2.2 Classification of splitting symmetries

For each of the 14 singularities, the Hermitian intersection form on $H_2(V_\star, \mathbb{C})$ is non-degenerate of positive signature 2. Our aim set in the introduction is to obtain equivariant monodromy groups M_χ which are hyperbolic reflection groups, that is, the restriction of the intersection form to the summand H_χ is non-degenerate and of positive signature 1. Hence the rank 2 positive subspace $H_+ \subset H_2(V_\star, \mathbb{C})$ must split between two character subspaces. We refer to a symmetry satisfying this condition as a *splitting symmetry*, and to the two characters as the *hyperbolic characters*. We will use this terminology even in the extreme situation, when the two H_χ are one-dimensional.

Lemma 2.1. *Assume symmetry g is quasihomogeneous. Then g is splitting if and only if $Jac(g) \notin \mathbb{R}$. In this case, the hyperbolic characters are $Jac(g)$ and its conjugate.*

Proof. According to [27], the rank 2 subspace in the cohomology $H^2(V_\star, \mathbb{C})$ dual to H_+ is spanned by the forms $\alpha = dx \wedge dy \wedge dz / dF_\star$ and $Hess(f)\,\alpha$. The two forms are eigenvectors of the automorphism $g^\star$ of $H^2(V_\star, \mathbb{C})$, with the eigenvalues $Jac(g)$ and its conjugate. $\square$

Corollary 2.2. *Non-quasihomogeneous exceptional unimodal functions have no splitting symmetries.*

Indeed, a symmetry of such a function preserves the modular term $Hess(f)$. Hence both α and $Hess(f)\,\alpha$ are in the same character subspace in the cohomology.

Lemma 2.3. *Assume a symmetry g of a quasihomogeneous exceptional unimodal function is a power of its Coxeter transformation: $g = C^p$. Then $Jac(g) = \varepsilon_N^{-p}$.*

Since $Jac(C) = \varepsilon_N^{w_x + w_y + w_z}$, this follows from the relation $w_x + w_y + w_z = N - 1$ which holds for all such singularities.

Our classificational result on normal forms of splitting symmetries, is

Theorem 2.4. *Any invariant splitting symmetry g of a quasihomogenous exceptional unimodal singularity f falls into one of the following categories.*

   a) The symmetry g of order $m > 2$ is a power of the Coxeter transformation C of function f.

   b) Each of the corank 2 singularities $E_{14}, Z_{13}, W_{13}, W_{12}$ admits symmetries g of order $m > 2$ which are powers of the Coxeter transformation composed with the involution $\iota_z(x, y, z) = (x, y, -z)$.

   c) Remaining symmetries are listed in Table 2.

Table 2 lists the symmetries up to a choice of a different generator of the same cyclic group.

Table 2: Special symmetries of Q_{12} and U_{12}

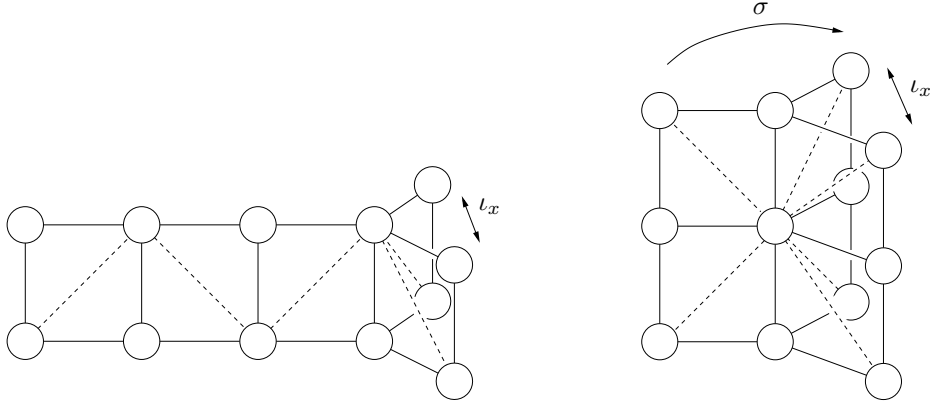| f	$g : x, y, z \mapsto$	g	$|g|$	g−codim
$Q_{12}: \ x^2 z + y^3 + z^5$	$\varepsilon_{10}^9 x, \omega y, \varepsilon_5 z$	$\iota_x C$	30	1
$\iota_x : (x, y, z) \mapsto$	$\varepsilon_{10}^7 x, y, \varepsilon_5^3 z$	$\iota_x C^3$	10	2
$\mapsto (-x, y, z)$	$-x, \omega^2 y, z$	$\iota_x C^5$	6	5
$U_{12}: \ x^3 + y^3 + z^4$	$\omega^2 x, y, iz$	σC	12	2
$\sigma : (x, y, z) \mapsto$	$x, \omega y, -z$	σC^2	6	4
$\mapsto (\omega x, \omega^2 y, z)$	$\omega x, \omega^2 y, -iz$	σC^3	12	2
	$\omega^2 x, y, z$	σC^4	3	6
$U_{12}: \ x^2 y + y^3 + z^4$	$\varepsilon_6^5 x, \omega y, iz$	$\iota_x C$	12	1
$\iota_x : (x, y, z) \mapsto$	$\varepsilon_6 x, \omega^2 y, -z$	$\iota_x C^2$	6	2
$\mapsto (-x, y, z)$	$-x, y, -iz$	$\iota_x C^3$	4	4
	$\varepsilon_6^5 x, \omega y, z$	$\iota_x C^4$	6	3

Figure 1: Symmetries of the Dynkin diagrams of Q_{12} (left) and U_{12} (right).

In Table 2, g-codim is the dimension of the base of a g-miniversal deformation.

According to Example 1.1, monomials to use in a g-miniversal deformation in case a) may be taken to be exactly those from Table 1 of weights divisible by the order m of the symmetry g. A similar choice in case b) coincides with that for the corresponding power of the Coxeter transformation. All these monomials along with possible choices in case c) are listed later, in Table 3. For the corank 2 functions not mentioned in part b), the symmetry $\iota_z : (x, y, z) \mapsto (x, y, -z)$ is $C^{N/2}$.

Theorem 2.4, in particular, states that, for any quasihomogeneous exceptional unimodal singularity f, we can make a quasihomogeneous coordinate change which diagonalises a splitting symmetry. In the case of U_{12}, there are two possible normal forms. This is similar to the two normal forms of the D_4 singularity.

The sign change in part b) of the Theorem is the $-id$ map on the vanishing homology. It does not affect the actual summands in the decomposition (2). It only affects the indexation, changing the signs of all characters.

The transformations ι_x and σ in Table 2 correspond to the order 2 and 3 symmetries of the Dynkin diagrams of the underlying singularities D_6 and D_4. The relevant symmetries of the Q_{12} and U_{12} Dynkin diagrams are shown in Figure 1 (the diagrams are constructed as those for the direct sums $D_6 \oplus A_2$ and $D_4 \oplus A_3$ of singularities, using the Gabrielov method [9]). Both ι_x and σ have real determinants, hence are able to split the subspace H_+ only in combination with a power of the Coxeter transformation which splits H_+ itself, that is, has order greater than 2.

Proof of the Theorem is rather straightforward and we shall only mention its steps. It starts with a diagonalisation of a symmetry which is a routine exercise on transformations of quasihomogeneous functions. After that we are reduced to consideration of diagonal symmetries $g : (x, y, z) \mapsto (ax, by, cz)$ of a trinomial function $\sum_{j=1,2,3} x^{t_{j1}} y^{t_{j2}} z^{t_{j3}}$, that is, to solutions of the system of monomial equations $a^{t_{j1}} b^{t_{j2}} c^{t_{j3}} = 1$, $j = 1, 2, 3$. For the normal forms from Table 1, the number $det(t_{jk})$ of such solutions is N in case a) of the Theorem, and $2N$ in case b). For the normal forms from Table 2, this number is either $2N$ or $3N$. For each function, N solutions are powers of the Coxeter transformation. The rest are products of such powers with respectively $\iota_z, \iota_x, \sigma, \sigma^2$. Finally, we use Lemmas 2.1 and 2.3 to ensure that the order of an involved power of the Coxeter element must be greater than 2.

3 Description of the hyperbolic monodromy groups

In this section we put together all the information sufficient to describe the action of the equivariant monodromy on the hyperbolic character subspaces singled out in the previous section. The information will be encoded into Dynkin diagrams.

3.1 Skeletons of Dynkin diagrams

First of all, each such diagram will contain a presentation of the corresponding generalised braid group, that is, of the fundamental group of $\Lambda \setminus \Sigma$. For this, we are using the standard method going back to

Zariski. We take a generic plane P in Λ, a generic line L in P, a generic base point $\star$ in L, choose a distinguished system of paths in L from $\star$ to points of $L \cap \Sigma$ and take the set of simple loops in L corresponding to the paths as generators of $\pi_1(\Lambda \setminus \Sigma, \star)$. These generators correspond to vertices of the diagram. The vertices are ordered following the counterclockwise order in which paths of the distinguished system leave the base point. However, all our diagrams will be trees, for which the order may be done arbitrary and hence omitted (see [18]).

Relations between the generators are read from the pair $(P, P \cap \Sigma)$: merger of two points – we never have more than two – of $L \cap \Sigma$ at a singular point of $P \cap \Sigma$ provides a *braiding relation* on the two generators, a and b, of the local monodromy group: $aba \ldots = bab \ldots$ with k factors either side if the singularity of $P \cap \Sigma$ is $\lambda_1^2 = \lambda_2^k$, in some local coordinates λ_1, λ_2 on P. The only possibilities we are meeting for the discriminants in our settings, are $k = 2, 3, 4, 6$. Respectively, the two vertices of the diagram representing the two generators will be joined by either no edge (the generators commute), or a simple, or a double or a triple edge.

The diagram obtained at this stage will be called the *skeleton* of the Dynkin diagram of the singularity with symmetry.

It turns out that, in all but two of our cases, the discriminant of a symmetric singularity coincides with the discriminant of one of the Weyl groups, hence for the skeleton of our Dynkin diagram we are able to take the standard Dynkin diagram of the group. The empirical rule to get the right skeleton is that the ratio of the weights of the parameters in a quasihomogeneous versal deformation should coincide with the ratio of the degrees of basic invariants of the related Weyl group. The two exceptional cases will be considered in section 3.4.

Each vertex of the diagram will be decorated outside with the singularity type of the relevant g-orbit of critical points, and with the self-intersection number of the corresponding vanishing χ-cycle.

Each edge will be decorated with the intersection number of the two vanishing cycles. Since all our diagrams are trees, it will not matter in which order we are intersecting the cycles. In the edge decoration we have certain freedom: due to the ambiguities in constructing vanishing χ-cycles, the intersection numbers are well-defined only up to multiplication by ± 1 and by powers of χ.

No edge between two vertices is equivalent to the intersection number of the cycles being zero.

3.2   The eigenvalue of a Picard-Lefschetz operator

The last data included in our Dynkin diagrams will be the orders of the Picard-Lefschetz operators which we will write inside the vertices. In fact each order r will be telling us the only non-trivial eigenvalue of the operator: on the $\chi = Jac(g)$ hyperbolic subspace the eigenvalue is ε_r.

Since the character $Jac(g)$ has a special role, we will use a special notation η for it.
Consider the cohomological direct sum

$$H^2(V_\star, \mathbb{C}) = \oplus_{\chi^m = 1} H^\chi, \tag{3}$$

where the substitution $g^\star$ is multiplication by χ on H^χ. Each summand H^χ here is dual to the summand H_χ in (2).

We have $\alpha = dx \wedge dy \wedge dz / dF_\star \in H^\eta$. The forms $\{\varphi_i \alpha\}$, where the $\{\varphi_i\}$ is the g-invariant part of a monomial basis of the local ring Q_f of f, form a basis of H^η. Therefore the g-codimension of the function f coincides with the dimension of H^η. Since the ring Q_f is Gorenstein, the g-codimension of f also coincides with the dimension of $H^{\overline{\eta}}$.

We observe that the subspace H^η is the only summand in (3) that contains a holomorphic nowhere-vanishing 2-form, α. This helps us to find the eigenvalues of the basic operators acting on H_η.

Proposition 3.1. *Consider the Picard-Lefschetz operator h_η on H_η corresponding to a g-orbit of critical points with a quasihomogeneous normal form $\psi(x', y', z')$. Choose the weights w_1', w_2', w_3' of the variables so that the weight of the function ψ is 1. Then the only eigenvalue of h_η distinct from 1 is $\exp(2\pi i (w_1' + w_2' + w_3'))$.*

Proof. The restriction of the family F to a line germ transversal to Σ may be brought near any of the critical points to a local normal form $\psi(x', y', z') + \epsilon$. Locally, the cohomological operator $h^\star = \oplus h^\chi$ is induced by a loop in $\mathbb{C}_\epsilon$ going once around the origin in the positive direction. Its eigenvectors are the

2-forms $\omega_j = \alpha_j(x', y', z')\, dx' \wedge dy' \wedge dz'/d\psi$, where the α_j form a monomial basis of the local ring of function ψ. The transformation $h^\star$ is the substitution $x' := \exp(2\pi i w_1')\, x'$ etc. Hence its eigenvalue on ω_j is $\exp(2\pi i\, weight(\omega_j))$, where $weight(\omega_j) = weight(\alpha_j) + w_1' + w_2' + w_3'$.

The only eigenform ω_j that vanishes nowhere in a neighbourhood of our elementary critical point is the one in which α_j is a non-zero constant, that is, has weight 0. $\qquad\square$

Corollary 3.2. *Assume a Picard-Lefschetz operator h_η on H_η corresponds to a g-orbit of simple critical points of type $X = A_k, D_k, E_k$. Then the only non-trivial eigenvalue of the operator h_η is $\varepsilon_{N'}$ where N' is the Coxeter number of the Weyl group X.*

This is so since for simple function singularities $w_1' + w_2' + w_3' = 1 + 1/N'$.

We recall the Coxeter numbers of the Weyl groups which we will need:

group	A_k	D_k	E_6	E_7
Coxeter number	$k+1$	$2(k-1)$	12	18

This means that the inner marking of the vertices by the order of the operators is excessive. However, we prefer to keep it.

3.3 The diagrams of the hyperbolic groups

Theorem 3.3. *For each exceptional unimodal function singularity with a splitting symmetry, there exists a distinguished basis of the hyperbolic subspace H_η in the vanishing homology, for which the Dynkin diagram is the one listed in the last column of Table 3 on page 9.*

On the $H_{\bar{\eta}}$, the monodromy representation is conjugate.

In Table 3, the symmetries g are given up to a choice of a different generator of the same cyclic group. The table also singles out g-invariant subsets of the monomial bases of the local rings from Table 1. These are monomials which may be used to construct a g-miniversal deformation of the singularity. For completeness, we have included g-codimension 1 singularities into the Table.

We have two kinds of diagrams in Table 3: standard (marked with $\star$) and non-standard. The latter ones are collected in Figure 3 on page 11: first of all we list 2-vertex diagrams in the order they appear in the Table, then similarly 3-vertex and finally 4-vertex diagrams. The inequality sign on an edge there is open towards the longer root, that is, towards the vanishing η-cycle corresponding to a longer orbit of critical points. In the notation of the diagrams of types B, F, G, the first upper index is the order of the Picard-Lefschetz operators corresponding to short roots, and the second upper index denotes the same for long roots. In the notation of the C type diagrams, the convention is opposite.

A $\star$ in the last column of the Table indicates that the Dynkin diagram may be derived from Figure 2 in the following standard way. The Dynkin diagrams $A_k^{(m)}$, $D_k^{(m)}$, $E_k^{(m)}$ have the usual ADE skeletons, while the labelling of their vertices and edges are exactly as those of the $A_2^{(m)}$ diagram. The Dynkin diagrams $B_k^{(m,m)}$, $C_k^{(m,m)}$, $F_4^{(m,m)}$, $G_2^{(m,m)}$ are foldings of the diagrams in the previous sentence in the usual manner, remembering to double (or triple) intersection numbers where two (or three) vertices or edges merge. The Dynkin diagram $C_k^{(2,m)}$ is constructed from $C_3^{(2,m)}$ by extending to the right with subdiagrams of type $A_2^{(m)}$.
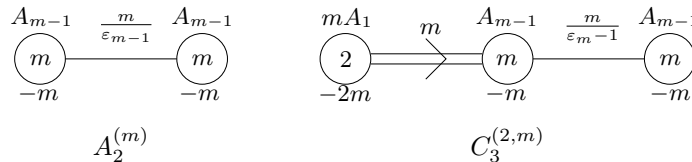

Figure 2: Standard Dynkin diagrams.

Table 3: Dynkin diagrams for invariant unimodal singularities with splitting symmetries g

f	g	$\lvert g\rvert$	versal monomials	diagram
E_{12} $x^3 + y^7 + z^2$	C, C^2	42,21	1	-
	C^3, C^6	14,7	$1, x$	$A_2^{(7)} \star$
	C^7, C^{14}	6,3	$1, y, y^2, y^3, y^4, y^5$	$A_6^{(3)} \star$
Z_{11} $x^3y + y^5 + z^2$	C, C^2	30,15	1	-
	C^3, C^6	10,5	$1, xy^2$	$G_2^{(10,2)} \star$
	C^5, C^{10}	6,3	$1, y, y^2, y^3, y^4$	$C_5^{(2,3)} \star$
E_{13} $x^3 + xy^5 + z^2$	C, C^2	30,15	1	-
	C^3, C^6	10,5	$1, x, y^5$	$C_3^{(2,5)} \star$
	C^5, C^{10}	6,3	$1, y^3, y^6, xy^2$	$E_{13}\lvert\mathbb{Z}_6$
Q_{10} $x^2z + y^3 + z^4$	C	24	1	-
	C^2	12	$1, z^2$	$B_2^{(12,3)}$
	C^3	8	$1, y$	$A_2^{(8)}$
	C^4	6	$1, z, z^2, z^3$	$C_4^{(3,3)} \star$
	C^6	4	$1, y, z^2, yz^2$	$F_4^{(4,2)} \star$
	C^8	3	$1, z, x, z^2, z^3$	$D_5^{(3)} \star$
E_{14} $x^3 + y^8 + z^2$	C	24	1	-
	C^2	12	$1, y^4$	$B_2^{(12,3)}$
	C^3	8	$1, x$	$A_2^{(8)} \star$
	C^4	6	$1, y^2, y^4, y^6$	$B_4^{(3,3)} \star$
	C^6	4	$1, x, y^4, xy^4$	$F_4^{(4,2)} \star$
	C^8	3	$1, y, y^2, y^3, y^4, y^5, y^6$	$A_7^{(3)} \star$
Z_{12} $x^3y + xy^4 + z^2$	C	22	1	-
W_{12} $x^4 + y^5 + z^2$	C	20	1	-
	C^2	10	$1, x^2$	$B_2^{(5,5)} \star$
	C^4	5	$1, x, x^2$	$A_3^{(5)} \star$
	C^5	4	$1, y, y^2, y^3$	$A_4^{(4)} \star$
Q_{11} $x^2z + y^3 + yz^3$	C, C^2	18,9	1	-
	C^3, C^6	6	$1, y, z^3$	$C_3^{(2,6)}$

f	g	$\lvert g\rvert$	versal monomials	diagram
Z_{13} $x^3y + y^6 + z^2$	C	18	1	-
	C^2	9	$1, y^3$	$B_2^{(18,3)}$
	C^3	6	$1, y^2, y^4$	$B_3^{(6,3)}$
	C^6	3	$1, y, y^2, y^3, y^4, y^5$	$C_6^{(2,3)} \star$
S_{11} $x^2z + yz^2 + y^4$	C	16	1	-
	C^2	8	$1, y^2$	$B_2^{(8,4)}$
	C^4	4	$1, y, y^2, z^2$	$C_4^{(2,4)}$
W_{13} $x^4 + xy^4 + z^2$	C	16	1	-
	C^2	8	$1, x^2$	$B_2^{(8,4)}$
	C^4	4	$1, x, x^2, y^4$	$C_4^{(2,4)} \star$
Q_{12} $x^2z + y^3 + z^5$	$C, \iota_x C$	15,30	1	-
	$C^3, \iota_x C^3$	5,10	$1, y$	$A_2^{(10)}$
	$\iota_x C^5$	6	$1, z, z^2, z^3, z^4$	$C_5^{(3,3)} \star$
	C^5	3	$1, z, x, z^2, z^3, z^4$	$D_6^{(3)} \star$
S_{12} $x^2z + yz^2 + xy^3$	C	13	1	-
U_{12} $x^3 + y^3 + z^4$	C	12	1	-
	σC	12	$1, y$	$A_2^{(12)}$
	C^2	6	$1, z^2$	$B_2^{(6,6)}$
	σC^2	6	$1, y, z^2, yz^2$	$F_4^{(3,3)} \star$
	C^3	4	$1, x, y, xy$	$D_4^{(4)} \star$
	σC^3	12	$1, xy$	$G_2^{(4,4)} \star$
	C^4	3	$1, z, z^2$	$A_3^{(6)}$
	σC^4	3	$1, z, z^2, yz, yz^2$	$E_6^{(3)} \star$
U_{12} $x^2y + y^3 + z^4$	$C, \iota_x C$	12	1	-
	$C^2, \iota_x C^2$	6	$1, z^2$	$B_2^{(6,6)}$
	$\iota_x C^3$	4	$1, y, y^2, xz^2$	$U_{12}\lvert\mathbb{Z}_4$
	C^3	4	$1, x, y, y^2$	$D_4^{(4)} \star$
	$C^4, \iota_x C^4$	6	$1, z, z^2$	$A_3^{(6)}$

Example 3.4. a) Consider the $D_6 \oplus A_2$ diagram from Figure 2.2. Its fusion along the A_2 direction (that is, replacement of each vertical A_2 subdiagram by a vertex – see [10] for details) yields the $D_6^{(3)}$ diagram. Its further folding by the involution ι_x delivers $C_5^{(3,3)}$.

On the other hand, the D_6 fusion of the $D_6 \oplus A_2$ diagram gives us the $A_2^{(10)}$ diagram from Figure 3. Its roots are of different length than in the case of the standard $A_2^{(10)}$ with the A_9 vertices (which comes as a result of the A_9 fusion of the $A_9 \oplus A_2$ diagram, that is, arises from the order 10 symmetry $(x, y, z) \mapsto (\varepsilon_{10}x, y, z)$ of the function $x^{10} + y^3 + z^2$). However, the rank 2 reflection groups defined by both standard and non-standard $A_2^{(10)}$ diagrams of course coincide.

b) Similarly, the A_3 fusion of the diagram $D_4 \oplus A_3$ from Figure 2.2 gives us $D_4^{(4,4)}$. The ι_x-folding of the latter is $C_3^{(4,4)}$, while its triple folding by the symmetry σ provides $G_2^{(4,4)}$.

The D_4 fusion of $D_4 \oplus A_3$ gives us the non-standard $A_3^{(6)}$ diagram from Figure 3, which defines the same reflection group as the standard diagram which one obtains by the A_5 fusion of the $A_5 \oplus A_3$ Dynkin diagram of $x^6 + y^4 + z^2$.

Proof of Theorem 3.3 is based on case-by-case calculations, and we give only its outline here. The details may be found in [16].

The proof starts with calculation of the discriminants of g-miniversal families. Only two of the discriminants are not of Weyl groups, and the details of all calculations in these two cases are in the next section.

The standard cases (marked $\star$ in the Table) are dealt with immediately – as it is shown in Example 3.4 – via fusion along the direct A_{m-1} summands in the relevant diagrams of the functions f, and, if needed, further double or triple folding.

In the non-standard cases rank 3 or 4 cases, the diagrams are constructed by using the adjacencies to the simpler singularities with similar cyclic symmetries whose diagrams have been obtained in [10, 11, 13, 14, 15].

In the non-standard rank 2 cases, the self-intersection numbers of the vanishing A and D χ-cycles are already known from the paper series just mentioned. For the E_6 η-cycle, such number is retrievable from [21]. For the order 9 symmetric E_7 η-cycle, we need some routine calculations based on consideration of the order 3 symmetry of E_7 (see [10]).

Once we know the self-intersection numbers and the eigenvalues of the Picard-Lefschetz operators, the braiding relation yields in a rank 2 case the absolute value of the intersection number of the two vanishing η-cycles. The number itself is an integer linear combination of powers of η, and the ambiguities in choices of the cycles already mentioned in Section 3.1 allow us to reduce it to the one given in Figure 3.

3.4 Two exceptional cases

3.4.1 $E_{13}|\mathbb{Z}_6$

A monomial invariant miniversal deformation of this singularity may be taken in the form

$$x^3 + xy^5 + z^2 + \delta xy^2 + \gamma y^6 + \beta y^3 + \alpha \,.$$

Here $\alpha, \beta, \gamma, \delta$ are the deformation parameters, and we can take

$$g = C^5 : (x, y, z) \mapsto (\overline{\omega}x, \overline{\omega}y, -z)$$

for the order 6 symmetry.

The calculations show that the discriminant consists of two irreducible components:

$$
\begin{aligned}
D_4 : \quad & \alpha = 0 \,, \\
3A_1 : \quad & 3125\alpha^3 - 729\beta^5 - 13500\beta\gamma^2\alpha^2 + 729\beta^4\delta\gamma + 729\beta^4\gamma^3 \\
& +16\delta^6\gamma^3 - 216\delta^3\beta^3 - 16d^6\beta + 16\delta^7\gamma + 216\delta^4\beta^2\gamma + 216\delta^3\beta^2\gamma^3 \\
& +4125\delta^2\gamma\alpha^2 - 5625\delta\beta\alpha^2 - 5832\beta^2\gamma^4\alpha + 6075\beta^3\gamma\alpha \\
& +2700\delta^2\beta^2\alpha + 864\delta^3\gamma^4\alpha + 888\delta^4\gamma^2\alpha + 16200\gamma^3\delta\alpha^2 - 5670\beta^2\gamma^2\delta\alpha \\
& -2592\delta^2\beta\gamma^3\alpha - 3420\delta^3\beta\gamma\alpha + 11664\gamma^5\alpha^2 + 16\delta^5\alpha = 0 \,.
\end{aligned}
$$

$G_2^{(10,2)}$

D_6 — 5 — $5A_1$
$10 \quad\quad 2$
$5(\sqrt{5}-3)/2 \quad\quad -10$

$B_2^{(12,3)}(Q_{10})$

E_6 — $2\sqrt{3}(1+i)$ — $2A_2$
$12 \quad\quad 3$
$2\sqrt{3}-6 \quad\quad -6$

$A_2^{(8)}$

D_5 — $4(1-\varepsilon_8)$ — D_5
$8 \quad\quad 8$
$4\sqrt{2}-8 \quad\quad 4\sqrt{2}-8$

$B_2^{(12,3)}(E_{14})$

E_6 — $4(1+\varepsilon_6)$ — $4A_2$
$12 \quad\quad 3$
$2\sqrt{3}-6 \quad\quad -12$

$B_2^{(18,3)}$

E_7 — $3(\omega-\varepsilon_9)(\omega-1)$ — $3A_2$
$18 \quad\quad 3$
$-9\left(1+\frac{\varepsilon_9^5-\varepsilon_9^7}{1-\omega}\right) \quad\quad -9$

$B_2^{(8,4)}(S_{11})$

D_5 — $4(1+i)$ — $2A_3$
$8 \quad\quad 4$
$4\sqrt{2}-8 \quad\quad -8$

$B_2^{(8,4)}(W_{13})$

A_7 — $4(1+i)(1+\varepsilon_8)$ — $2A_3$
$8 \quad\quad 4$
$-8 \quad\quad -8$

$A_2^{(10)}$

D_6 — 5 — D_6
$10 \quad\quad 10$
$5(\sqrt{5}-3)/2 \quad\quad 5(\sqrt{5}-3)/2$

$A_2^{(12)}$

E_6 — $2\sqrt{3}(1+i)$ — E_6
$12 \quad\quad 12$
$2\sqrt{3}-6 \quad\quad 2\sqrt{3}-6$

$B_2^{(6,6)}$

D_4 — 6 — $2D_4$
$6 \quad\quad 6$
$-3 \quad\quad -6$

$C_3^{(2,6)}$

$3A_1$ — 3 — D_4 — 3 — D_4
$2 \quad\quad 6 \quad\quad 6$
$-6 \quad\quad -3 \quad\quad -3$

$B_3^{(3,6)}$

A_5 — 6 — $2A_2$ — $2(1-\omega)$ — $2A_2$
$6 \quad\quad 3 \quad\quad 3$
$-6 \quad\quad -6 \quad\quad -6$

$A_3^{(6)}$

D_4 — 3 — D_4 — 3 — D_4
$6 \quad\quad 6 \quad\quad 6$
$-3 \quad\quad -3 \quad\quad -3$

$E_{13}|\mathbb{Z}_6$

$3A_1$ — 3 — D_4 — 3 — $3A_1$ — 3 — $3A_1$
$2 \quad\quad 6 \quad\quad 2 \quad\quad 2$
$-6 \quad\quad -3 \quad\quad -6 \quad\quad -6$

$F_4^{(4,2)}(Q_{10})$

A_3 — $2(1+i)$ — A_3 — $2\sqrt{2}$ — $2A_1$ — 2 — $2A_1$
$4 \quad\quad 4 \quad\quad 2 \quad\quad 2$
$-4 \quad\quad -4 \quad\quad -4 \quad\quad -4$

$F_4^{(4,2)}(E_{14})$

A_3 — $2(1+i)$ — A_3 — 4 — $4A_1$ — 4 — $4A_1$
$4 \quad\quad 4 \quad\quad 2 \quad\quad 2$
$-4 \quad\quad -4 \quad\quad -8 \quad\quad -8$

$C_4^{(2,4)}$

$2A_1$ — $2(1+i)$ — A_3 — $2(1+i)$ — A_3 — $2(1+i)$ — A_3
$2 \quad\quad 4 \quad\quad 4 \quad\quad 4$
$-4 \quad\quad -4 \quad\quad -4 \quad\quad -4$

$U_{12}|\mathbb{Z}_4$

$2A_1$ — $2(1+i)$ — A_3 — $2(1+i)$ — A_3 — 4 — $4A_1$
$2 \quad\quad 4 \quad\quad 4 \quad\quad 2$
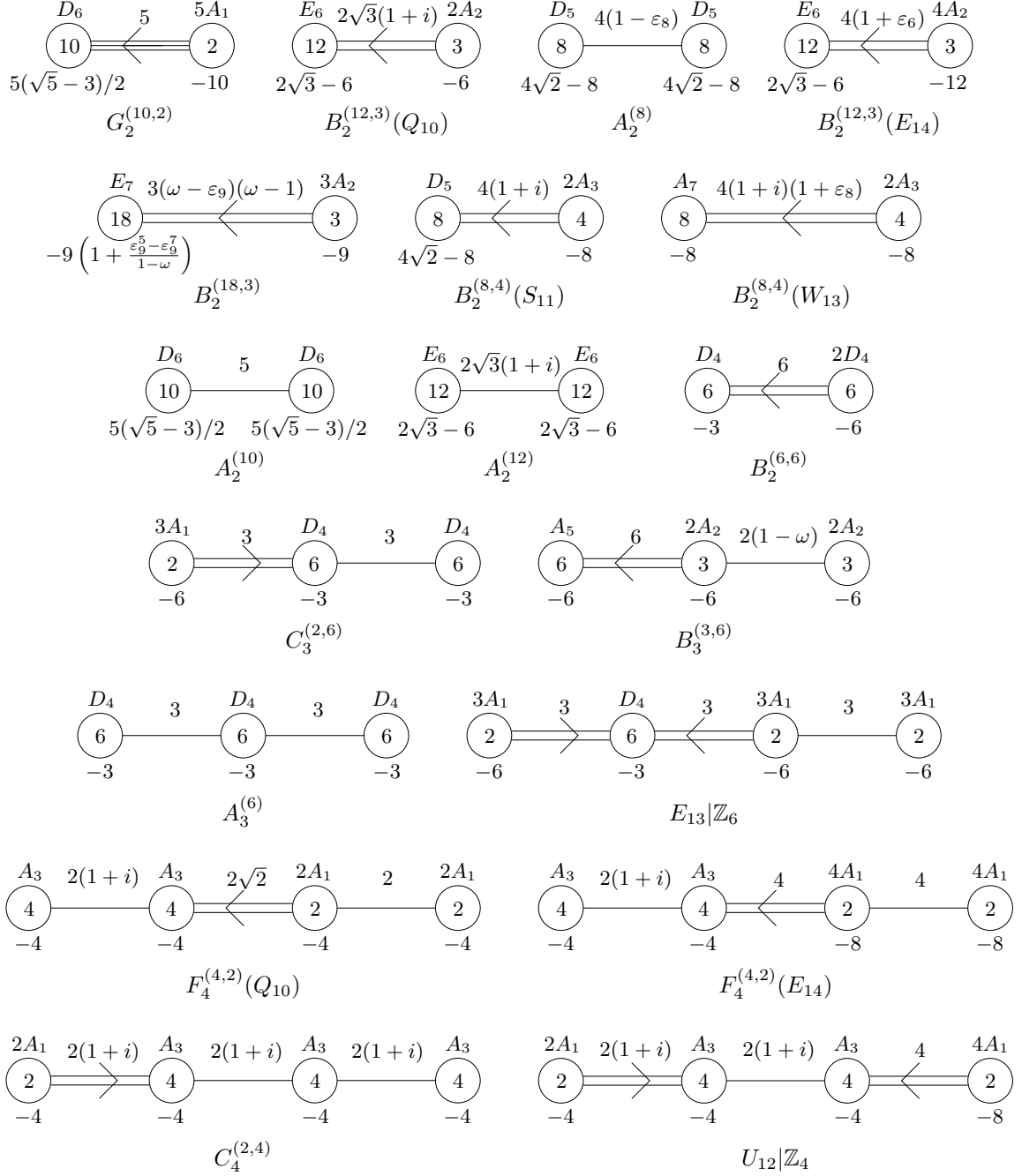$-4 \quad\quad -4 \quad\quad -4 \quad\quad -8$

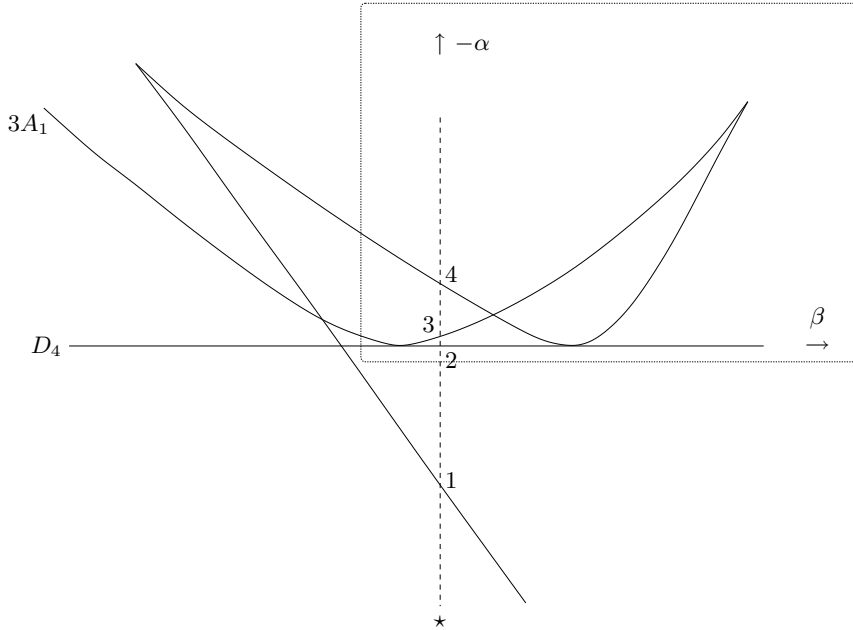Figure 3: The non-standard Dynkin diagrams for the hyperbolic groups arising in our classification.

Figure 4: Inclusion of a generic planar section of the $J_{10}|\mathbb{Z}_6$ discriminant (boxed) into that of $E_{13}|\mathbb{Z}_6$.

In particular, the discriminant is different from the irreducible discriminant of the Shephard-Todd group G_{29} [23]. This is in spite of the coincidence of the ratio $(6:12:18:30)$ of the weights of the deformation parameters with the ratio $(4:8:12:20)$ of the degrees of basic invariants of the group.

A generic section of the discriminant is shown in Figure 4, taken with sufficiently large $\gamma = \delta = const < 0$. The figure emphasises the adjacency

$$E_{13}|\mathbb{Z}_6 \to J_{10}|\mathbb{Z}_6,$$

which allows us to use the $J_{10}|\mathbb{Z}_6$ Dynkin diagram as a building block for our hyperbolic case.

For a generic line in the base of our invariant deformation we take the dashed line in Figure 4. For a distinguished system of paths in this line joining the base point $\star$ with the discriminantal points, we use arcs in the half-plane $\mathrm{Im}(-\alpha) \geq 0$. This orders the discriminantal points as indicated in the Figure. Such choice yields the Dynkin diagram below.
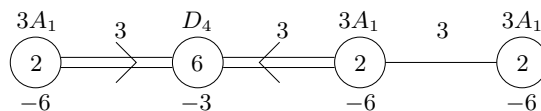
Here the generators are h_3, h_2, h_4, h_1 from left to right. The subdiagram on the three left vertices is that of $J_{10}|\mathbb{Z}_6$ (see [13] where the singularity is called $J_{10}|\mathbb{Z}_3$ since the symmetry there preserves z). According to Figure 4, the generator h_1 commutes with the subgroup generated by h_2 and h_3, and satisfies $h_1 h_4 h_1 = h_4 h_1 h_4$. The latter implies that the first vanishing cycle may be chosen so that its intersection number with the forth is 3.

3.4.2 $U_{12}|\mathbb{Z}_4$

This time we take an invariant miniversal deformation in the form

$$x^2 y + y^3 + z^4 + \delta x z^2 + \gamma y^2 + \beta y + \alpha,$$

and the symmetry is $g = \iota_x C^3 : (x,y,z) \mapsto (-x, y, -iz)$. The ratio $(2:4:8:12)$ of the weights of the deformation parameters does not repeat the ratio of the degrees of basic invariants of any Shephard-Todd group.

The discriminant has three irreducible components:

$$
\begin{aligned}
2A_1 : \quad & \alpha = 0\,, \\
A_3 : \quad & (27\alpha - 9\beta\gamma + 2\gamma^3)^2 + 4(3\beta - \gamma^3)^3 = 0\,, \\
4A_1 : \quad & \delta^6 + 4\gamma\delta^4 + 16\beta\delta^2 + 64\alpha = 0\,.
\end{aligned}
$$

The union of the first two components is the B_3 (equivalently C_3) discriminant multiplied by the δ-axis.

To construct a generic two dimensional section of the discriminant shown in Figure 5, we took $\gamma = const < 0$, and tilted slightly by setting $\delta = \gamma + \epsilon\beta$, for some fixed small $\epsilon > 0$. In the Figure, the A_3 component is displayed in bold for distinction, and the dashed line is a generic line which provides the generators. A distinguished system of paths in this line consists of four arcs in $\mathrm{Im}(-\alpha) \leq 0$ joining the base point $\star$ with the discriminantal points.
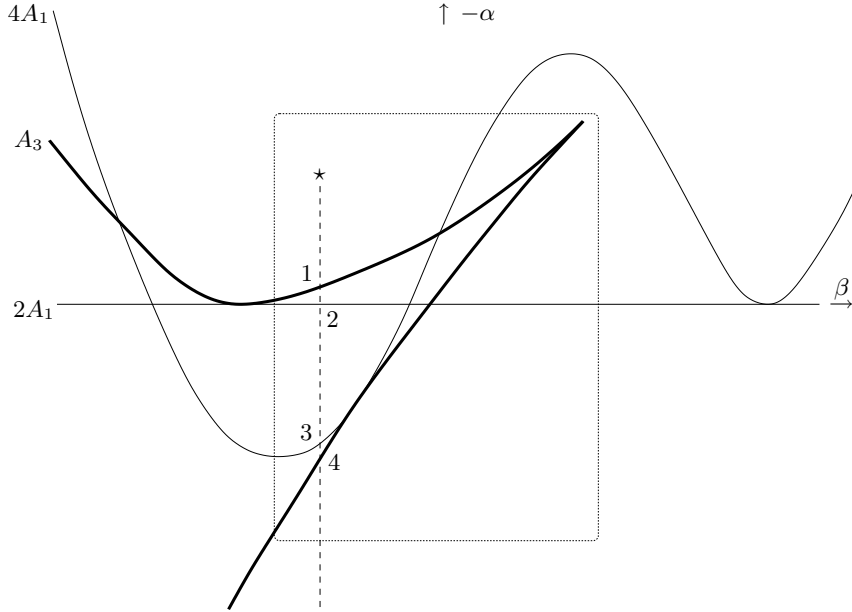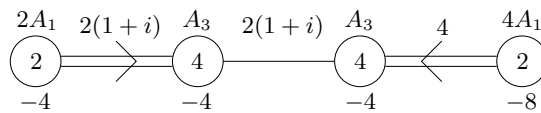

Figure 5: Generic section of $U_{12}|\mathbb{Z}_4$ discriminant.

Let us show that the path choice yields the Dynkin diagram

in which the generators from left to right are h_2, h_1, h_4, h_3. We first notice that the adjacency

$$U_{12} \to J_{10} : \quad x^2 y + y^3 + (tx + z^2)^2\,, \quad t \in \mathbb{C}\,,$$

is compatible with the symmetry and hence provides an invariant adjacency

$$U_{12}|\mathbb{Z}_4 \to J_{10}|\mathbb{Z}_4\,.$$

The latter singularity was studied in [13] under the name $C_3^{(4)}$. Such notation emphasised that its discriminant is isomorphic to the C_3 discriminant, and one can take for its Dynkin diagram for the characters $\pm i$ the above diagram with the leftmost vertex omitted. Moreover, consistently with that, a generic planar section of the $J_{10}|\mathbb{Z}_4$ discriminant is diffeomorphic to the one boxed in Figure 5 with the component $2A_1$ omitted.

Now Figure 5 tells us that the operator h_2 commutes with the subgroup generated by h_3 and h_4, while $(h_1 h_2)^2 = (h_2 h_1)^2$. The last relation, the self-intersection of the second vanishing cycle being -4, and the intersection number of the first two vanishing cycles being a Gaussian number confirms that the first cycle may be chosen so that this intersection number is $2(1 + i)$.

13

4 Projectivised rank 2 groups

Assume coordinates $z_0, \ldots, z_k$ in the space $\mathbb{C}^{k+1}$ equipped with a hyperbolic Hermitian form are chosen so that the form is $-|z_0|^2 + |z_1|^2 + \cdots + |z_k|^2$. The group $U(k,1) \subset GL(k+1, \mathbb{C})$ sends the cone

$$\mathcal{C} = \{-|z_0|^2 + |z_1|^2 + \cdots + |z_{k-1}|^2 < 0\}$$

into itself. Respectively, in the chart $z_0 \neq 0$ of $\mathbb{CP}^k$, the projective group $PU(k,1)$ acts on the ball

$$\{|w_1|^2 + \cdots + |w_k|^2 < 1\} \subset \mathbb{C}^k, \quad w_j = z_j / z_0, \quad j = 1, \ldots, k.$$

The ball is the standard model for the complex hyperbolic k-space. In particular, for $k = 1$, this is the Poincaré disk $\mathbb{H}$.

For a triple of positive integers $r_1 \leq r_2 \leq r_3$ such that $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} < 1$, there is a triangle Δ in $\mathbb{H}$ with angles π/r_1, π/r_2, π/r_3, which is unique up to isometry. The hyperbolic reflections in the sides of the triangle generate a group $D(r_1, r_2, r_3)$ of isometries of $\mathbb{H}$, which has the triangle Δ as its fundamental domain.

We refer to the index 2 subgroup $D_+(r_1, r_2, r_3) \subset D(r_1, r_2, r_3)$ consisting of holomorphic transformations as a *triangle group*. Its fundamental domain Δ_+ is the union of two adjacent copies of Δ.

Theorem 4.1. *The projectivisations $\mathbb{P}M_\eta$ of all rank 2 monodromy groups contained in Table 3 are triangle groups $D_+(r_1, r_2, r_3)$. Table 4 identifies all such groups.*

Table 4: Triangle Groups

singularities (f, g)	$|g|$	M_η	r_1, r_2, r_3
(E_{12}, C^3)	14	$A_2^{(7)}$	$2, 3, 7$
(Z_{11}, C^3)	10	$G_2^{(10,2)}$	$2, 3, 10$
$(Q_{10}, C^2), (E_{14}, C^2)$	12	$B_2^{(12,3)}$	$2, 3, 12$
$(Q_{10}, C^3), (E_{14}, C^3)$	8	$A_2^{(8)}$	$2, 3, 8$
(W_{12}, C^2)	10	$B_2^{(5,5)}$	$2, 5, 5$
(Z_{13}, C^2)	9	$B_2^{(18,3)}$	$2, 3, 18$
$(S_{11}, C^2), (W_{13}, C^2)$	8	$B_2^{(8,4)}$	$2, 4, 8$
(Q_{12}, C^3)	5	$A_2^{(10)}$	$2, 3, 10$
$(U_{12}, \sigma C)$	12	$A_2^{(12)}$	$2, 3, 12$
(U_{12}, C^2)	6	$B_2^{(6,6)}$	$2, 6, 6$
$(U_{12}, \sigma C^3)$	12	$G_2^{(4,4)}$	$3, 4, 4$

The rule for the r_i is as follows. The orbit space of the skeleton Weyl group is a weighted homogeneous $\mathbb{C}^2$ isomorphic to the base of a g-miniversal deformation of the related symmetric singularity. If the Weyl group is A_2, the weight ratio is $2 : 3$, which gives us two of the r_i, the third being the order of a Picard-Lefschetz operator corresponding to the only components of the discriminant. In the B_2 and G_2 cases, the weight ratios $1 : 2$ and $1 : 3$ give us one of the r_i, while the other two are the orders of Picard-Lefschetz operators corresponding to the two discriminantal components.

This suggests a general Looijenga-type (cf. [17])

Conjecture 4.2. *Let $\Lambda = \mathbb{C}^{k+1}$ be the base of a g-miniversal deformation of an invariant singularity from Table 3, and $\Sigma \subset \Lambda$ the discriminant of the singularity. Let Λ' be the quotient of the hyperbolic cone $\mathcal{C} \subset \mathbb{C}^{k+1}$ by the monodromy group M_η, and $\Sigma' \subset \Lambda'$ the set of irregular orbits. Then the pairs (Λ, Σ) and (Λ', Σ') are biholomorphic.*

The Conjecture, in particular, asserts that the complex hyperbolic reflection groups M_η are discrete.

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