Doubling Hypercuboids

Joel A. Haddley

1 Introduction

In this work we examine the problem of integral hypercuboids for which the internal core and the external shell are of equal volume. That is, the volume of the entire hypercuboid is double that of the internal shell. We will call such a hypercuboid a doubling hypercuboid. The problem is well known in two and three dimensions. In this article, we have extended the treatment to include higher (and lower) dimensions.

We have used a computer program to solve the program in four and five dimensions. We show that the number of solutions in any given dimension is finite, and give some general results about the smallest and largest possible solutions.

Finally, we analyse our computer program to show that it has some limitations when it comes to finding solutions in arbitrary dimensions.

2 Background

Martin Gardner in his textbook "Wheels, Life and other Mathematical Amusements" [1] describes a well known mathematical puzzle for schoolchildren:

> *Find a rectangle composed of unit squares for which the interior and border comprise the same number of squares, and hence have the same area.*

We will call such a rectangle a *doubling rectangle*. The situation is described by the equation

$$a_1 a_2 = 2(a_1 - 2)(a_2 - 2), \tag{1}$$

where a_1, a_2 are the lengths of the edges. Gardener reported that Longley-Cook had solved the problem by rearranging Equation 1 to get

$$(a_1 - 4)(a_2 - 4) = 8, \tag{2}$$

and noticing that $(a_1 - 4)$ and $(a_2 - 4)$ must be positive factors of 8. Up to reordering, the only two such pairs are $2, 4$ and $1, 8$ giving the only two doubling rectangles:

$$(a_1, a_2) = (6, 8), \text{ or } (a_1, a_2) = (5, 12).$$

These are shown in Figure 1.

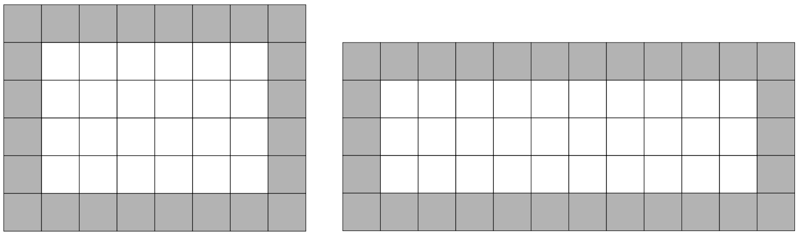

Figure 1: The doubling rectangles $(6, 8)$ and $(5, 12)$

Extending the problem to three dimensions Greenblatt [2] restates the problem. Take a cuboid with integral side lengths (a_1, a_2, a_3). Paint the outside, then cut up the cuboid into unit cubes. If the number of cubes with paint on is equal to the number without, we will call the cuboid a *doubling cuboid.* This is equivalent to finding all positive integer solutions to

$$a_1a_2a_3 = 2(a_1 - 2)(a_2 - 2)(a_3 - 2). \tag{3}$$

Greenblatt used an intuitive approach to find *ten* of the possible solutions, and noted (with complete accuracy, as it turned out) that were "about ten more solutions." According to Gardener, the complete list of the 20 solutions was found by Sleator using a computer program. Gardener presented the smallest and largest solutions (the 'size' of a solution will soon be defined precisely) as

$$(a_1, a_2, a_3) = (8, 10, 12), \text{ and } (a_1, a_2, a_3) = (5, 13, 132)$$

respectively. These are shown to relative scale in Figure 2.

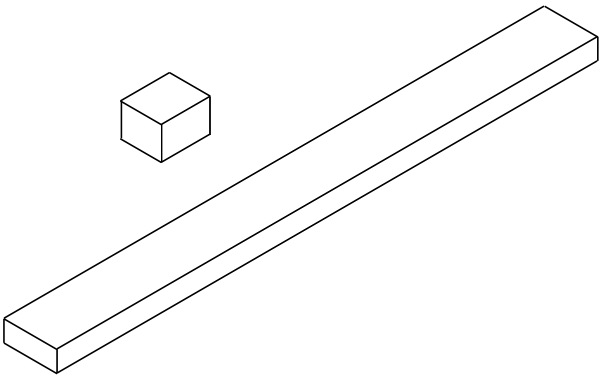

Figure 2: The doubling cuboids $(8, 10, 12)$ and $(5, 13, 132)$.

3 Doubling Hypercuboids

In this section, we give some results about the generalisation of this problem into other dimensions (note: we are careful no to say 'higher dimensions'). We define a *doubling hypercuboid* to be a hypercuboid with integral side lengths $(a_1, a_2, \ldots, a_n)$ that satisfy the generalised equation

$$\prod_{i=1}^{n} a_i = 2\prod_{i=1}^{n}(a_i - 2), \tag{4}$$

where $3 \leq a_i$ for all i so that the number of interior hypercubes is non-zero. It will also be convenient to express Equation 4 in terms of base 2 logarithms:

$$\sum_{i=1}^{n} \log_2 \frac{a_i}{a_i - 2} = 1. \tag{5}$$

Since $f(x) = \log_2 \frac{x}{x-2}$ is a strictly decreasing function for $x > 2$ (i.e. $y > x \Rightarrow f(y) < f(x)$), we deduce the stronger condition $4 \leq a_i$ since $f(3) = \log_2 3 > 1$ is too large a summand. In order to avoid repetition we shall impose an ordering on the side lengths. A reordering corresponds only to a repositioning of the hypercuboid. We assume

$$4 \leq a_1 \leq a_2 \leq \cdots \leq a_n. \tag{6}$$

Proposition 3.1. *The number of doubling hypercuboids in dimension n is finite.*

Proof. Since $f(x) = \log_2 \frac{x}{x-2}$ is a strictly decreasing function for $x > 2$, the effect of increasing any a_i will corresponding to decreasing the corresponding summand in Equation 5. To maintain equality, another summand must be increased which in turn corresponds to decreasing some a_j, $j \neq i$. Since a_j is bounded below according Equation 6, a_i must be bounded above. □

Since there are finitely many solutions we may discuss maximum and minimum solutions. We therefore must define these notions. The solution for which a_n takes its maximum value will be called the *maximum solution*. According to the proof of Proposition 3.1, steadily increasing a_n will cause all a_i, $i \neq n$, to steadily decrease. Thus in particular a_1 will achieve its lowest possible value in the maximum solution.

We will called the solution for which a_1 achieves its maximum value the *minimum solution*. Similarly, a_n will achieve its lowest possible value in the minimum solution.

In this sense, we think of the maximum and minimum solutions as measuring how spread out the solutions are. Think of the solutions as being the bellows of an accordion. The maximum solution corresponds to the bellows being expanded, the minimum to them being compressed.

3.1 The Maximum Solution

The main result of this section is the following.

Theorem 3.2. *Define the iterative sequence $(m_k)_{k\geq 1}$ as*

$$m_1 = 4, \quad m_{k+1} = m_k(m_k - 1) \tag{7}$$

Then the maximum solution in dimension n is given by

$$(a_1, \ldots, a_n), \text{ where } a_i = \begin{cases} m_n & \text{if } i = n, \\ 1 + m_i & \text{otherwise.} \end{cases} \tag{8}$$

The sequence $(m_k)_{k\geq 1}$ is sequence A204321 in [3]. Before the formal proof, we will explain the intuition behind this theorem.

- When $n = 1$, there is only one solution

  $$m_1 = 4,$$

  which is automatically the maximum solution.

- When $n = 2$, we can find the maximum solution by modifying the maximum solution for $n = 1$. Increasing the previous value for m_1 by the smallest possible amount means the contribution to Equation 5 coming from the summand corresponding to m_2 will be as small as possible, and since the summands are strictly decreasing functions we maximise m_2 and therefore achieve the maximum solution. So we take $1 + m_1 = 5$ and solve Equation 4 in order to find $(1 + m_1, m_2) = (5, 12)$.

- When $n = 3$, we again look to achieve our maximum by modifying the previous case, $n = 2$. We increase the values of m_1 and m_2 by an amount to make their corresponding contributions to Equation 5 decreased by the smallest possible amount. Thus we add 1 to m_2 and do not modify m_1. This will guarantee that m_3 is as large as possible when solving Equation 4. We find $(1 + m_1, 1 + m_2, m_3) = (5, 13, 132)$.

So given a maximum solution $(1+m_1, \ldots, 1+m_{n-1}, m_n)$ in dimension n, we look for a maximum solution in dimension $n+1$ of the form $(1+m_1, \ldots, 1+m_n, m_{n+1})$ by solving Equation 4 to find m_{n+1} in terms of the given $m_1, \ldots, m_n$. The following proof shows that this equation always has a positive integer solution and derives the iterative function.

Proof. For $n = 1$ the only solution to Equation 4 is $a_1 = 4$. We assume that the theorem is true for all $n \leq k$, for some fixed k. That is, the maximum solution in dimension k is known and is

$$(1 + m_1, \ldots, 1 + m_{k-1}, m_k). \tag{9}$$

Since this is a solution of Equation 4, it satisfies

$$\frac{m_k}{m_k - 2} \prod_{i=1}^{k-1} \frac{m_i + 1}{m_i - 1} = 2. \tag{10}$$

Following the heuristic argument preceding this proof we modify this solution to find a solution for $n = k + 1$. That is, we try

$$(1 + m_1, \ldots, 1 + m_{k-1}, 1 + m_k, m_{k+1}). \tag{11}$$

Substituting in to Equation 4 gives

$$\frac{m_{k+1}}{m_{k+1} - 2} \prod_{i=1}^{k} \frac{m_i + 1}{m_i - 1} = 2. \tag{12}$$

Using Equation 10 as our inductive hypothesis, Equation 12 may be cancelled to give

$$m_{k+1} = m_k(m_k - 1)$$

as required. To conclude our proof, we need only remark that since m_{k-1} is a positive integer, m_k must also be a positive integer. □

Table 1 gives the first few terms in the sequence $(m_k)_{k\geq 1}$. This sequence grows at an extremely large rate. Moreover, due to the similarity with well studied, often chaotic, logistic map (see [4], for example) we have no expectation to find a closed form for m_k in terms of k alone. We can see that it doesn't take long at all before the size of m_k becomes unmanageable.

Table 1: Values of iteratively defined sequence $(m_k)_{k\geq 1}$

k	m_k
1	4
2	12
3	132
4	17292
5	$298, 995, 972$
6	$\approx 8.9 \times 10^{16}$

3.2 The Minimum Solution

We are not aware of any direct methods to find the minimum solution in a given dimension, so we will give bounds on the components of this solution. Let us denote the minimum solution by $(w_1, \ldots, w_n)$.

Proposition 3.3. *For dimension $n \geq 2$ there is no solution $(a_1, \ldots, a_n)$ such that $a_i = a_j$ for all i, j.*

Proof. Assume otherwise for a contradiction. Let $a_i = \alpha$ for all i, where α is a positive integer. Then Equation 4 gives

$$\alpha^n = 2(\alpha - 2)^2,$$

or equivalently

$$\alpha = 2\frac{2^{1/n}}{2^{1/n} - 1}.$$

Then α is irrational and, in particular, is not a positive integer. □

Had such a solution existed, this would surely have been the minimum solution.

Proposition 3.4. *For dimension $n \geq 2$, there is a unique solution satisfying*

$$a_{i+1} = a_i + 2 \text{ for all } i = 1, \ldots, n - 1,$$

namely

$$a_i = 2(n + i), \quad (i = 1, \ldots, n).$$

Proof. This is an automatic consequence of substituting in to Equation 4 and cancelling. □

Propositions 3.3 and 3.4 immediately give the following corollary. We observe that this corollary also holds when $n = 1$, in which case the solution is unique and therefore both the maximum and minimum at the same time.

Corollary 3.5. *The minimum solution* $(w_1, \ldots, w_n)$ *satisfies*

$$\begin{aligned} 2(n+1) \leq \quad & w_1 \quad \leq \left\lfloor 2\frac{2^{1/n}}{2^{1/n}-1} \right\rfloor, \\ \left\lceil 2\frac{2^{1/n}}{2^{1/n}-1} \right\rceil \leq \quad & w_n \quad \leq 4n \end{aligned}$$

where $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$ is the standard floor function defined by $\lfloor x \rfloor = \max\{z \in \mathbb{Z} : z \leq x\}$, $\lceil \cdot \rceil : \mathbb{R} \to \mathbb{Z}$ is the standard ceiling function defined by $\lceil x \rceil = \min\{z \in \mathbb{Z} : z \geq x\}$.

Knowing these bounds decreases the computational complexity of finding minimum solutions. These are presented in Table 2 for the first few dimensions, along with the predicted lower and upper bounds on w_1 and w_n.

Table 2: Minimum solution by dimension

n	Bounds on w_1	Bounds on w_n	Minimum solution
1	$[4,4]$	$[4,4]$	4
2	$[6,6]$	$[7,8]$	$(6,8)$
3	$[8,9]$	$[10,12]$	$(8,10,12)$
4	$[10,12]$	$[13,16]$	$(11,12,13,15)$
5	$[12,15]$	$[16,20]$	$(14,14,16,16,18)$
6	$[14,18]$	$[19,24]$	$(16,17,18,19,20,21)$

4 Computational Results

In Table 3 we present all solutions in dimensions where there are sufficiently few solutions to do so. For dimensions 4 and 5 we state the total number of solutions.

Table 3: Solutions in low dimensions

n	Solutions					
1	4					
2	(6,8) (5,12)					
3	20 solutions:	$(8,10,12)$	$(8,9,14)$	$(7,10,16)$	$(8,8,18)$	$(6,14,16)$
		$(7,9,20)$	$(6,12,20)$	$(6,11,24)$	$(7,8,30)$	$(6,10,32)$
		$(5,22,24)$	$(5,20,27)$	$(5,18,32)$	$(5,17,36)$	$(5,16,42)$
		$(6,9,56)$	$(5,15,52)$	$(5,14,72)$	$(7,7,100)$	$(5,13,132)$
4	374 solutions					
5	21313 solutions					

These explicit lists of results were found using a Maple program which is organised as follows. The variable a_1 loops through all possible values. A sub-loop sends a_2 through all possible values, and so on. Once a nested loop has values for $(a_1, \ldots, a_n)$ this is checked against Equation 4 to check whether it is a solution or not. We let

$$D = d_1(a_1, \ldots, a_n) = a_1 + \cdots + a_n$$

be the standard d_1 metric on $\mathbb{R}^n$ (modulus signs are omitted since in our case a_i is a positive integer for all i), and D is always a positive integer. According to Theorem 3.2 and Corollary 3.5 we have bounds

$$2n(n+1) \le D \le (n-1) + \sum_{i=1}^{n} m_i, \tag{13}$$

so we use D as our parent loop parameter. This is rather convenient as there is no necessity to find all solutions in one attempt, we can instead set up our program to run over a subinterval for D and repeat as necessary, perhaps on several computers in parallel, until we have covered the entire range.

To find the local bounds on the component a_k, thought of as a loop parameter in our program, we note that its parent loop parameters are fixed. These are $a_1, \dots, a_{k-1}$ and D. Then the minimum value a_k can take is a_{k-1} according to our ordering criterion. As a_k gets larger, a_j ($j > k$) get smaller according to the proof of Proposition 3.1. Therefore let $\alpha \ge a_{k-1}$ be a real number such that $(a_1, \dots, a_{k-1}, \alpha, \dots, \alpha)$ such that Equation 4 is satisfied. Then $a_k \le \alpha$. But since our attention is restricted to the case when D is constant, finding α is elementary. We have the local bounds on the loop parameter a_k as

$$a_{k-1} \le a_k \le \lfloor \alpha \rfloor,$$

where

$$\alpha = \frac{1}{n-k}\left(D - \sum_{i=1}^{k-1} a_i\right).$$

We recall the $a_1, \dots, a_{k-1}$ and D are fixed as loop parameters of parent loops.

Counting these loops for a fixed D is the same as counting the number of restricted partitions of D (see [5], for example). Assume $n \ge 2$. The number of partitions of D into exactly n parts with each part at least 5 is the same as the number of partitions of $D' = D - 5n$ into at most n parts. This in turn is the same as the number of partitions of D' such that each part is at most n. According to Equation 13, we have

$$D_{min} = n(2n-3) \le D' \le (-4n-1) + \sum_{i=1}^{n} m_i = D_{max}$$

Let $p_n(d)$ denote the number of partitions of d such that each part is at most n. Then the number of loops $\ell(n)$ in our program is given by

$$\ell_n = \sum_{d=D_{min}}^{D_{max}} p_n(d).$$

For $2 \le n \le 4$ the value of $\ell(n)$ can be worked out directly using generating series. For $n \ge 5$ the large numbers involved are difficult to compute directly, so we use the asymptotic approximation

$$p_n(d) \approx \frac{d^{n-1}}{n!(n-1)!}$$

due to [6]. This approximation is excellent in the cases where it can be verified.

Table 4:

n	$\ell(n)$
2	18
3	73761
4	1.6×10^{14}
5	1.7×10^{38}
6	9.8×10^{95}

Table 4 gives the value of $\ell(n)$ for $2 \leq n \leq 6$.
The solutions in 1, 2 and 3 dimensions were known. We were able to extend the classification to include all solutions in 4 and 5 dimensions, as well as giving some general results on the extreme solutions. However, our methods for listing all solutions in a given dimension are not sufficient in dimension $n \geq 6$. For such high dimensions, alternative methods must be sought.

References

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